SYMMETRIES AND EXACT SOLUTIONS OF NONLINEAR DIRAC EQUATIONS

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Physical law should have
mathematical beauty

P.-M. Dirac

PREFACE

The Dirac equation describing motion of an elementary particle with spin 1/2 (electron or proton) is an inseparable part of the modern mathematical and theoretical physics. Together with the Maxwell and Schrödinger equations it forms a basis of the quantum mechanics, quantum electrodynamics and quantum field theory.

Following Dirac's discovery of the linear equation of an electron there appeared fundamental papers by D.D. Ivanenko, W. Heisenberg, R. Finkelstein with collaborators and F. Gürsey advocating the idea of nonlinear description of an elementary particle with spin 1/2 which made it possible to take into account its self-interaction. Furthermore, W. Heisenberg put forward the idea to use a nonlinear Dirac equation as a possible basis model for a unified field theory. These ideas have contributed substantially to the modern view of an elementary particle as a complex dynamical system described (modeled) by a nonlinear system of partial differential equations. The general structure of such nonlinear equations is determined by the Lorentz-Poincaré-Einstein or the Galilei relativity principle.

Till now there is no book devoted to a systematic study of nonlinear generalizations of the classical Dirac equation. So it was our primary intention to write a book devoted entirely to a comprehensive and detailed group-theoretical study of first–order nonlinear spinor partial differential equations satisfying either the Lorentz–Poincaré–Einstein or the Galilei relativity principle. These equations contain, as particular cases, the nonlinear spinor models suggested by D.D. Ivanenko, W. Heisenberg, R. Finkelstein and F. Gürsey.

In the course of research we have discovered that the methods and techniques developed to study nonlinear Dirac equations can be successfully applied to a wide range of Poincaré- and Galilei-invariant nonlinear multi-dimensional equations of modern quantum field theory describing interactions of spinor, scalar and vector fields.

As a result, the book has a 'two-level' structure. At the first level, it may be considered as a self-contained group-theoretical introduction to the theory of the

first-order nonlinear spinor equations with a particular emphasis on a development of efficient methods for constructing their exact (classical) solutions. At the second level, we employ these methods to construct multi-parameter families of exact solutions of nonlinear wave, Dirac-d'Alembert, Maxwell-Dirac, d'Alembert-eikonal, $SU(2)$ Yang-Mills, Lévy-Leblond, and some other partial differential equations. Furthermore, the approach used enables us to give a systematic and unified treatment of the related questions such as conditional symmetry of differential equations, separation of variables in linear systems of partial differential equations, and integrability of some nonlinear systems of differential equations in two-independent variables.

It was our aim to write a book in a form accessible not only for 'pure theoreticians' but also for those who are interested in applications of group-theoretical/symmetry methods to concrete nonlinear systems of partial differential equations. Every opportunity is taken to illustrate general statements by specific examples and to reduce to a reasonable minimum the level of abstractness in the exposition.

The book is based on the authors' results obtained at the Institute of Mathematics of the National Academy of Sciences of Ukraine in 1984–1996 [139, 140, 146], [148]–[171], [291]–[320]. It also accumulates a rich experience of other groups working in the related areas of group-theoretical, algebraic-theoretical analysis of differential equations. The bibliography is claimed to be the most comprehensive and complete as far as symmetry and exact solutions of nonlinear spinor equations are concerned. But it is not our intention to give the full list of references devoted to application and development of group-theoretical methods in the mathematical and theoretical physics. Only references used directly are cited.

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INTRODUCTION

In 1913 the outstanding French mathematician Elie Cartan discovered spinors [42, 45]. He made this discovery while investigating irreducible representations of the group of rotations in the n-dimensional Euclidean space. He was the first to find and to describe in full detail spinor representations of the group of rotations.

The theory of spinors became an inseparable part of mathematical and theoretical physics after Dirac's discovery of the equation of motion for an electron (1928) which bears his name [69, 71]. The four complex-valued functions of four arguments contained in the Dirac equation are the components of a spinor with respect to the Lorentz group.

It is interesting to note that the methods used by Cartan and Dirac to discover spinors are essentially different. These methods lie at the basis of algebraic-theoretical, group-theoretical investigations in modern quantum theory.

Today spinors and spinor representations play a basic role in mathematical and theoretical physics, since all elementary particles, classical and quantum fields having half-integer spins ($s = 1/2, 3/2, 5/2, \ldots$) are described with their help. Moreover, using de Broglie's heuristic idea of "fusion" we can construct particles (fields) having integer spins ($s = 0, 1, 2, \ldots$) from a particle (field) having the spin $s = 1/2$. That is why the theory of spinors and spinor analysis as the principal analytical apparatus for investigation of spinor dynamical systems are useful in solving problems from other fields of mathematics and quantum physics.

The first paper devoted to a nonlinear generalization of the Dirac equation was published by Ivanenko in 1938 [192]. Later Finkelstein with collaborators in 1951 [80, 81] and Heisenberg in 1953 [180, 181] started analyzing various nonlinear generalizations of the Dirac equation.

Heisenberg [181]–[184] put forward a vast program on the construction of a

unified field theory of elementary particles. As a basis of this theory he chose a self-interacting spinor field described by a nonlinear equation. According to Heisenberg such a field is determined by the following Dirac-type nonlinear equation:

$$i\gamma_\mu \partial_\mu \psi + \lambda \gamma_\mu \gamma_4 (\bar{\psi}\gamma^\mu \gamma_4 \psi)\psi = 0, \tag{0.1}$$

where ψ is a four-component Dirac spinor and λ is a parameter. We will call system (0.1) the Dirac-Heisenberg equation.

The present book deals with the following two principal problems: the first one is to describe systems of nonlinear spinor partial differential equations of the first and second orders invariant under the Poincaré and the Galilei groups and under their natural extensions; the second problem is the construction in explicit form of exact solutions of the classical nonlinear spinor, vector and scalar differential equations describing interaction of the Dirac, Maxwell and Yukawa fields.

Unlike the majority of researchers we do not derive nonlinear equations within the framework of the variational principle. We apply the symmetry selection principle, namely, from the whole set of partial differential equations (PDEs) of a given order we select those on whose sets of solutions some fixed representation of the Poincaré or the Galilei group is realized. Such an approach to the derivation of motion equations seems to be more general than the traditional method based on the Lagrange function [116, 119, 137].

The major part of the book is devoted to the development of efficient methods designed to obtain exact solutions of nonlinear equations. All these methods are based on the idea of reducing multi-dimensional partial differential equations to equations having smaller dimensions.

While reducing PDEs a key role is played by substitutions of the special form [88, 89, 92, 137, 155]

$$\psi(x) = A(x)\varphi(\omega_1, \omega_2, \ldots, \omega_n), \tag{0.2}$$

where $\varphi(\omega)$ is an unknown function-column and $A(x)$ is a variable matrix of corresponding dimensions; $\omega_\alpha = \omega_\alpha(x)$, $\alpha = 1, \ldots, n$ are real-valued scalar functions.

Explicit forms of the functions $A(x)$, $\omega_\alpha(x)$ are obtained by requiring that substitution of the expression (0.2) into the PDE under study reduces it to an equation containing only "new" dependent (φ) and independent ($\omega_1, \omega_2, \ldots, \omega_n$) variables. Of course, the availability of an effective procedure of computing the matrix $A(x)$ and the variables $\omega_\alpha(x)$ providing the

reduction of the initial equation is implied. Furthermore, the construction described above will be called the Ansatz for field $\psi(x)$.

Provided the equation under study possesses nontrivial local symmetry, there exists an effective algorithm for constructing Ansätze (suggested and applied for the first time to some of the simplest PDEs by Sophus Lie). Ansätze obtained in this way will be called Lie Ansätze.

In [91, 92] we suggested the generalization of the Lie method. The idea of this generalization is based on the following observation: the Lie method of constructing particular solutions, apart from its group-theoretical foundations, can be considered as addition of some first-order PDE to a given equation. Within the Lie approach this additional equation is a linear combination of basis elements of the invariance algebra of the equation under investigation. In view of this fact it was suggested to consider the coefficients of that linear combination as arbitrary functions of x, ψ, ψ_{x_μ}, $\psi_{x_\mu x_\nu}$. In other words the additional constraint on the set of solutions of the equation under investigation is, generally speaking, a nonlinear first- or second-order PDE with variable coefficients. Such a generalization proved to be constructive. In many cases it provided the possibility of obtaining broad classes of exact solutions of nonlinear equations which could not be found within the framework of the classical Lie approach [96, 97], [105]–[107], [120, 124, 108, 126, 127, 128, 137, 143], [154]–[160], [246, 303, 308].

With the use of nonlocal and conditional symmetry of linear and nonlinear spinor equations (the notion of the conditional symmetry of differential equations was introduced in [91, 116, 137]) we obtain wide classes of non-Lie Ansätze, which reduce these equations to systems of ordinary differential equations (ODEs).

Due to large symmetry of equations being considered systems of ODEs obtained by reduction via Lie and non-Lie Ansätze are often integrable by quadratures. Their exact solutions, after being substituted into the corresponding Ansätze, give rise to particular solutions of the nonlinear spinor equations under study.

As shown in Section 2.6, exact solutions of nonlinear spinor equations make it possible to construct exact solutions of other Poincaré-invariant equations. In particular, we construct a number of exact solutions of the nonlinear d'Alembert equation via solutions of the nonlinear Dirac equation.

More detailed information concerning the contents of the book is provided by chapter and section titles.

For the reader's convenience, we give a brief account of some facts, termi-

nology and notations from group theory which are used in the book (for more details, see [6, 33, 34, 41, 49, 76, 79, 190, 218, 233, 236]).

An r-parameter Lie transformation group G_r is a set of transformations of the space $\mathbb{R}^n \times \mathbb{C}^m$

$$\begin{aligned} x'_\alpha &= f_\alpha(x,u,\theta), \quad \alpha = 0,\dots,n-1, \\ u'_\beta &= g_\beta(x,u,\theta), \quad \beta = 0,\dots,m-1, \end{aligned} \tag{0.3}$$

$\theta \in U$, U is an open sphere in $\mathbb{R}^r$, where f_α and g_β are real-analytical functions of θ satisfying the following relations:

1. $f_\alpha(x,u,0) = x_\alpha, \quad g_\beta(x,u,0) = u_\beta,$
2. $\forall\{\theta_1,\theta_2\} \subset U,\ \exists\theta_3 = T(\theta_1,\theta_2) \in U:$
$f_\alpha\Big(f(x,u,\theta_1),\, g(x,u,\theta_1),\, \theta_2\Big) = f_\alpha(x,u,\theta_3),$
$g_\beta\Big(f(x,u,\theta_1),\, g(x,u,\theta_1),\, \theta_2\Big) = g_\beta(x,u,\theta_3).$

Here $T: U \times U \to U$ is a vector-function whose components are real-analytical functions satisfying the relations

1. $T(\theta,0) = T(0,\theta) = \theta, \quad \forall\theta \in U,$
2. $\forall\theta \in U,\ \exists\theta^{-1} \in U: \quad T(\theta,\theta^{-1}) = T(\theta^{-1},\theta) = 0,$
3. $\forall\{\theta_1,\theta_2,\theta_3\} \subset U: \quad T\Big(T(\theta_1,\theta_2),\theta_3\Big) = T\Big(\theta_1, T(\theta_2,\theta_3)\Big).$

The r-parameter Lie transformation group (0.3) is related to the r-dimensional vector space AG_r whose basis elements are first-order differential operators

$$Q_\tau = \sum_{\alpha=0}^{n-1} \xi_{\tau\alpha}(x,u)\frac{\partial}{\partial x_\alpha} + \sum_{\beta=0}^{m-1} \eta_{\tau\beta}(x,u)\frac{\partial}{\partial u_\beta}, \tag{0.4}$$

the coefficients $\xi_{\tau\alpha},\ \eta_{\tau\beta}$ being defined by the following formulae:

$$\begin{aligned} \xi_{\tau\alpha}(x,u) &= \left.\frac{\partial f_\alpha}{\partial\theta_\tau}\right|_{\theta=0}, \\ \eta_{\tau\beta}(x,u) &= \left.\frac{\partial g_\beta}{\partial\theta_\tau}\right|_{\theta=0}. \end{aligned} \tag{0.5}$$

The vector space AG_r is closed with respect to the operation

$$(X,Y) \to Z = XY - YX \equiv [X,Y]$$

and, consequently, forms an r-dimensional Lie algebra. This algebra is called the Lie algebra of the group G_r.

Conversely, given the Lie algebra with basis elements (0.4), where $\xi_{\tau\alpha}$ and $\eta_{\tau\beta}$ are sufficiently smooth functions, then the r-parameter Lie transformation group is obtained by solving the Lie equations

$$\begin{aligned} \frac{\partial f_\alpha}{\partial \theta_\tau} &= \xi_{\tau\alpha}(f,g), \quad f_\alpha(x,u,0) = x_\alpha, \\ \frac{\partial g_\beta}{\partial \theta_\tau} &= \eta_{\tau\beta}(f,g), \quad g_\beta(x,u,0) = u_\beta, \quad \tau = 1,\ldots,r \end{aligned} \tag{0.6}$$

and by constructing the superposition of the resulting one-parameter Lie groups.

Thus, there exists a one-to-one correspondence between a Lie transformation group G_r and its Lie algebra AG_r. To emphasize this correspondence we say that operators Q_τ generate the group G_r. These operators are called infinitesimal operators (generators) of the group G_r (as a rule, we omit the word "infinitesimal").

We say that the differential equation

$$L\Big(x, u(x)\Big) = 0 \tag{0.7}$$

is invariant under the group of transformations G_r (or: admits the group G_r) if the change of variables (0.3) transforms the set of solutions of equation (0.7) into itself. The group G_r is called invariance or symmetry group of equation (0.7). A corresponding Lie algebra is called invariance or symmetry algebra of the equation in question.

According to Lie [218] the differential equation (0.7) is invariant under the group G_r having generators (0.4) if and only if

$$\widetilde{Q}_\tau L \,\Big|_{[L]} = 0, \tag{0.8}$$

where $[L]$ means the set of solutions of the equation $L = 0$ and $\widetilde{Q}_\tau$ is the N-th prolongation of the operator Q_τ (N is the order of differential equation (0.7)).

The N-th prolongation of the operator

$$Q = \sum_{\alpha=0}^{n-1} \xi_\alpha(x,u)\frac{\partial}{\partial x_\alpha} + \sum_{\beta=0}^{m-1} \eta_\beta(x,u)\frac{\partial}{\partial u_\beta}$$

is constructed as follows

$$\widetilde{Q} = Q + \zeta_{\beta\alpha_1}\frac{\partial}{\partial\left(\dfrac{\partial u_\beta}{\partial x_{\alpha_1}}\right)} + \cdots + \zeta_{\beta\alpha_1\ldots\alpha_N}\frac{\partial}{\partial\left(\dfrac{\partial^N u_\beta}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_N}}\right)},$$

where

$$\zeta_{\beta\alpha_1} = D_{\alpha_1}\eta_\beta - \frac{\partial u_\beta}{\partial x_\alpha}D_{\alpha_1}\xi_\alpha,$$

$$\zeta_{\beta\alpha_1\alpha_2} = D_{\alpha_2}\zeta_{\beta\alpha_1} - \frac{\partial^2 u_\beta}{\partial x_{\alpha_1}\partial x_\alpha}D_{\alpha_2}\xi_\alpha,$$

$$\zeta_{\beta\alpha_1\ldots\alpha_N} = D_{\alpha_N}\zeta_{\beta\alpha_1\ldots\alpha_{N-1}} - \frac{\partial^N u_\beta}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_{N-1}}\partial x_\alpha}D_{\alpha_N}\xi_\alpha,$$

$$D_\alpha = \frac{\partial}{\partial x_\alpha} + \frac{\partial u^\beta}{\partial x_\alpha}\frac{\partial}{\partial u^\beta} + \sum_{n=1}^{\infty}\frac{\partial^{n+1}u_\beta}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_n}\partial x_\alpha}\;\frac{\partial}{\partial\left(\dfrac{\partial^n u_\beta}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_n}}\right)}$$

(summation over repeated indices is implied).

The invariance criterion (0.8) gives rise to a linear system of PDEs (the determining equations) for the functions ξ_α, η_β, whose general solution determines the maximal (in Lie sense) invariance algebra of the equation considered. The corresponding Lie group is called the maximal invariance (symmetry) group of equation (0.7).

The procedure described above is just Lie method for investigating symmetries of differential equations. Application of this method to equations of mathematical physics requires the performing of cumbersome computations (this is especially the case for multi-component systems of PDEs). If we deal with a system of linear PDEs

$$L(x)u(x) = 0, \quad u = (u_0, u_1, \ldots, u_{m-1})^T, \tag{0.9}$$

the computations can be substantially simplified. A symmetry operator acting in the linear space of solutions of system (0.9) is seeked in the form

$$Q = \sum_{\mu=0}^{n-1}\xi_\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x), \tag{0.10}$$

where $\xi_\mu(x)$ are smooth real-valued scalar functions, $\eta(x)$ is some $(m\times m)$-matrix. Within the Lie approach operator (0.10) is represented in the form

$$X = \sum_{\alpha=0}^{n-1}\xi_\alpha(x)\frac{\partial}{\partial x_\alpha} - \sum_{\beta_1,\beta_2=0}^{m-1}\eta_{\beta_1\beta_2}(x)u_{\beta_2}\frac{\partial}{\partial u_{\beta_1}}.$$

The invariance criterion for system of PDEs (0.9) reads (see, e.g., [115])

$$LQu(x)\Big|_{[Lu]} = 0. \tag{0.11}$$

Condition (0.11) means that the operator Q transforms the set of solutions of (0.9) into itself.

Relation (0.11) is rewritten in the following equivalent form:

$$[L, Q] = R(x)L, \tag{0.12}$$

where $R(x)$ is some $(m \times m)$-matrix. The above operator equality is to be understood in the following way: operators on the left- and right-hand sides of (0.12) give the same result when acting on an arbitrary solution of system (0.9).

Let us emphasize that the invariance algebra obtained by solving relation (0.11) or (0.12) is not the maximal one because any system of linear PDEs admits the Lie transformation group

$$\begin{aligned} x'_\mu &= x_\mu, \quad \mu = 0, \dots, n-1, \\ u'_\beta &= u_\beta + \theta u_{0\beta}(x), \quad \beta = 0, \dots, m-1, \end{aligned}$$

where θ is a real parameter, $u_0(x)$ is an arbitrary solution of the system considered. But the above Lie group gives no essential information about the structure of solutions of the equation under study and is not considered in the present book.

For many symmetry groups of systems of PDEs of mathematical and theoretical physics, the matrix $\eta(x)$ possesses very important algebraic properties which simplify substantially all manipulations with symmetry operators (0.10). Moreover, in most of the problems considered in this book we use the algebraic relations which are satisfied by $\eta(x)$, but we do not use their explicit form. That is why we will represent the infinitesimal symmetry operators in the form (0.10) (if it is possible and does not lead to confusion).

In the approach based on the formulae (0.10), (0.11) the restrictions of Lie method are quite evident since an operator transforming the set of solutions of equation (0.9) into itself does not have to be of the form (0.10) (a symmetry operator may belong to the class of differential operators of the order $N \geq 1$ or to the class of integro-differential operators [115, 116, 118]).

Below we give a list of notations and conventions used throughout the book.

A scalar product in the Minkowski space $R(1,3)$ with the metric tensor

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1,2,3 \end{cases}$$

is denoted by $a \cdot b = g_{\mu\nu} a_\mu b_\nu, \ \{a,b\} \subset R(1,3)$.

A scalar product in the Euclidean space $R(3)$ with the metric tensor $\delta_{ab} = -g_{ab}$ is written as follows

$$\vec{n} \cdot \vec{m} = \delta_{ab} \ n_a m_b = n_a m_a.$$

Summation over repeated indices is used, indices being denoted by the Greek letters α, β, μ, ν with the values 0, 1, 2, 3 and indices being denoted by the Latin letters a, b, c with the values 1, 2, 3 (unless otherwise indicated).

By the symbol ε_{abc} the antisymmetric tensor of rank three

$$\varepsilon_{abc} = \begin{cases} 1, & (a,b,c) = \text{cycle}\,(1,2,3), \\ -1, & (a,b,c) = \text{cycle}\,(2,1,3), \\ 0, & \text{in other cases} \end{cases}$$

is designated.

All the functions considered in the book are supposed to be differentiable as many times as is necessary. The derivative of a function of one variable $f = f(z)$ is denoted by a dot over the symbol $\dot{f} \equiv df/dz$. To distinguish a partial derivative we use the symbol ∂_z, i.e. $\partial f/\partial z \equiv \partial_z f$, and the partial derivative with respect to the μ-th independent variable is denoted by $\partial_\mu f \equiv \partial_{x_\mu} f$.

Vector and tensor indices are written as subscripts ($x_\mu, A_\mu, F_{\mu\nu}$, etc.) and spinor indices as superscripts (ψ^α). Lowering or raising of an index in the Minkowski space $R(1,3)$ is carried out by the metric tensor $g_{\mu\nu}$, for example,

$$x^\mu = g_{\mu\nu} x_\nu = \begin{cases} x_0, \ \mu = 0, \\ -x_a, \ \mu = a = 1,2,3. \end{cases}$$

Complex conjugation is denoted by the asterisk $(x+iy)^* = x - iy$ and the matrix transposed with respect to a given matrix A is designated by A^T. The symbol $A^\dagger$ stands for a complex conjugate of a transposed matrix, i.e. $(A^T)^* = A^\dagger$.

SYMMETRY OF NONLINEAR SPINOR EQUATIONS

The first chapter is of an introductory character. Here we present well-known facts about different representations of the Dirac equation [115, 116, 118], its local and nonlocal (non-Lorentz) symmetry and conservation laws for the Dirac field. Detailed group-theoretical analysis of nonlinear generalizations of the Dirac equation which are invariant under the Poincaré group $P(1,3)$, extended Poincaré group $\widetilde{P}(1,3)$ and conformal group $C(1,3)$ is carried out. Some second-order Poincaré- and conformally-invariant spinor equations are considered. Wide classes of nonlinear PDEs for spinor, scalar and vector fields invariant under the groups $P(1,3)$, $\widetilde{P}(1,3)$, $C(1,3)$ are described.

We establish correspondence between reducibility of PDEs and their conditional symmetry (the results obtained play a crucial role when constructing exact solutions of multi-dimensional partial differential equations).

1.1. Local and nonlocal symmetry of the Dirac equation

The Dirac equation is the system of four linear complex partial differential equations

$$(i\gamma_\mu\partial_\mu - m)\psi(x) = 0, \tag{1.1.1}$$

where $\psi = \psi(x_0, x_1, x_2, x_3)$ is the four-component, complex-valued function-column, $m = \text{const}$, γ_μ are (4×4)-matrices satisfying the Clifford-Dirac algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}I, \quad \mu,\nu = 0,\dots,3, \tag{1.1.2}$$

where I is the unit (4×4)-matrix.

Under the massless Dirac equation we mean system (1.1.1) with $m = 0$.

Since on the set of solutions of the Dirac equation a spinor representation of the Lorentz group is realized (see the Appendix 1), the function $\psi(x)$ is called the spinor field (or, for brevity, the spinor) and equation (1.1.1) as well as its nonlinear generalizations are called spinor equations.

If we act with the operator $i\gamma_\mu\partial_\mu + m$ on the left-hand side of equality (1.1.1) and use relations (1.1.2), then a system of four splitting wave equations for the spinor $\psi(x)$

$$(\partial_\mu\partial^\mu + m^2)\psi(x) = 0 \tag{1.1.3}$$

is obtained.

It is worth noting that Dirac derived equation (1.1.1) by factorizing the second-order differential operator $\partial_\mu\partial^\mu + m^2$, i.e., by representing it in the form of the product of two first-order operators $Q_\pm = i\gamma_\mu\partial_\mu \pm m$, whence it followed that γ_μ were matrices satisfying the algebra (1.1.2) [35, 69, 71].

1. Algebra of the Dirac matrices. We say that a representation of the Clifford-Dirac algebra is given if there are four (4×4)-matrices satisfying relations (1.1.2). There exist infinitely many representations of the Clifford-Dirac algebra. But all these representations are equivalent, namely, for each two sets of matrices $\left\{\gamma'_\mu\right\}$, $\left\{\gamma_\mu\right\}$ satisfying (1.1.2) there exists such a nonsingular (4×4)-matrix V that

$$\gamma'_\mu = V\gamma_\mu V^{-1}, \quad \mu = 0, \dots, 3. \tag{1.1.4}$$

If it is not indicated otherwise, we assume that the matrices γ_μ realize the following representation of the algebra (1.1.2):

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \tag{1.1.5}$$

where I, 0 are the unit and zero (2×2)-matrices, σ_a are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1.6}$$

In addition, we use the following representations of the Clifford-Dirac algebra:

$$\gamma_0 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -iI \\ -iI & 0 \end{pmatrix}; \tag{1.1.7}$$

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3. \tag{1.1.8}$$

Straightforward verification shows that the matrix $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$ satisfies relations of the form

$$\gamma_4\gamma_\mu + \gamma_\mu\gamma_4 = 0, \quad \gamma_4^2 = -1, \quad \mu = 0, \ldots, 3.$$

Matrices γ_0, γ_1, γ_2, γ_3, γ_4 form the maximal set of generators of the Clifford-Dirac algebra in the class of (4×4)-matrices.

The maximal set of generators of the Clifford-Dirac algebra in the class of (8×8)-matrices is exhausted up to the equivalence relation (1.1.4) by the following matrices:

$$\tilde{\Gamma}_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}, \qquad \tilde{\Gamma}_4 = \begin{pmatrix} 0 & \gamma_4 \\ \gamma_4 & 0 \end{pmatrix},$$
$$\tilde{\Gamma}_5 = \begin{pmatrix} 0 & i\gamma_4 \\ -i\gamma_4 & 0 \end{pmatrix}, \quad \tilde{\Gamma}_6 = \begin{pmatrix} \gamma_4 & 0 \\ 0 & -\gamma_4 \end{pmatrix}, \quad \mu = 0, \ldots, 3, \tag{1.1.9}$$

where 0 is the zero (4×4)-matrix.

It is known that all possible products of matrices γ_μ form a basis in the linear space of (4×4)-matrices. The elements of this basis can be chosen as follows

$$I, \quad \gamma_\mu, \quad \gamma_\mu\gamma_\nu, \quad \gamma_4\gamma_\mu, \quad \gamma_4, \quad \mu < \nu, \quad \mu, \nu = 0, \ldots, 3. \tag{1.1.10}$$

Sixteen matrices (1.1.10) are linearly independent and, consequently, an arbitrary (4×4)-matrix is represented as a linear combination of the basis elements (1.1.10).

2. Various formulations of the Dirac equation. The four-component function-row $\bar{\psi}(x) = (\psi(x))^\dagger\gamma_0$ is called a Dirac-conjugate spinor. To obtain

an equation for $\bar{\psi}(x)$ we apply a complex conjugation procedure to (1.1.1) with subsequent transposition and multiply the obtained expression by γ_0 on the right. Taking into account relations $\gamma_0^\dagger = \gamma_0$, $\gamma_a^\dagger = -\gamma_a$, we have

$$i\partial_\mu \bar{\psi}\gamma_\mu + m\bar{\psi} = 0. \tag{1.1.11}$$

If we designate

$$\tilde{\psi} = i\gamma_2\psi^*, \tag{1.1.12}$$

then equation (1.1.11) can be rewritten in the form

$$(i\gamma_\mu\partial_\mu - m)\tilde{\psi} = 0.$$

Hence it follows that system (1.1.1), (1.1.11) can be represented in the form of the eight-component equation

$$(i\widetilde{\Gamma}_\mu\partial_\mu - m)\Psi(x) = 0, \tag{1.1.13}$$

where

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \tilde{\psi}(x) \end{pmatrix}.$$

If we choose the matrices γ_μ in the representation (1.1.7), we can rewrite the Dirac equation (1.1.1) as a system of eight real PDEs

$$(i\widetilde{\Gamma}_\mu - m)\widetilde{\Psi}(x) = 0, \tag{1.1.14}$$

where

$$\widetilde{\Psi}(x) = \begin{pmatrix} \operatorname{Re}\psi(x) \\ \operatorname{Im}\psi(x) \end{pmatrix}.$$

On multiplying equation (1.1.1) by the matrix γ_0 on the left we get the Dirac equation in the Hamilton form

$$i\partial_0\psi = H\psi = (-i\gamma_0\gamma_a\partial_a + m\gamma_0)\psi.$$

Choosing the matrices γ_μ in the representation (1.1.8) and representing the spinor $\psi(x)$ in the form

$$\psi(x) = \begin{pmatrix} \varphi_-(x) \\ \varphi_+(x) \end{pmatrix}, \tag{1.1.15}$$

where $\varphi_\pm(x)$ are two-component functions, we rewrite the Dirac equation as follows

$$\begin{aligned} &(i\partial_0 + i\sigma_a\partial_a)\varphi_+ - m\varphi_- = 0, \\ &(i\partial_0 - i\sigma_a\partial_a)\varphi_- - m\varphi_+ = 0. \end{aligned} \tag{1.1.16}$$

Acting on the first equation of system (1.1.16) by the operator $i\partial_0 - i\sigma_a\partial_a$ we have

$$(\partial_\mu\partial^\mu + m^2)\varphi_+(x) = 0$$

and what is more $\varphi_-(x) = m^{-1}(i\partial_0 + i\sigma_a\partial_a)\varphi_+(x)$. Consequently, the system of four first-order differential equations (1.1.1) is equivalent to the system of two splitting wave equations.

From (1.1.16) it is clear that the massless Dirac equation

$$i\gamma_\mu\partial_\mu\psi(x) = 0 \tag{1.1.17}$$

splits into two Weyl equations for two-component spinors $\varphi_\pm(x)$.

Let us also note that the massless Dirac equation (1.1.17) can be represented in the form of the Maxwell equations with currents. To become convinced of this fact we represent the four-component function $\psi(x)$ in the following equivalent form:

$$\psi = \begin{pmatrix} -E_1 \\ E_3 \\ -H_2 \\ F \end{pmatrix} + i \begin{pmatrix} E_2 \\ G \\ -H_1 \\ H_3 \end{pmatrix}, \tag{1.1.18}$$

where $E_a,\ H_a,\ F,\ G$ are some smooth real-valued functions.

Substituting (1.1.18) into (1.1.17) and splitting with respect to i we get the Maxwell equations with currents [138]

$$\begin{aligned} \partial_0\vec{E} &= \operatorname{rot}\vec{H} + \vec{j}, \quad \operatorname{div}\vec{E} = j_0, \\ \partial_0\vec{H} &= -\operatorname{rot}\vec{E} + \vec{k}, \quad \operatorname{div}\vec{H} = k_0, \end{aligned} \tag{1.1.19}$$

where $j_\mu = \partial_\mu F,\ k_\mu = \partial_\mu G$.

The above presented formulations of the Dirac equation are, of course, equivalent but choosing an appropriate one we can substantially simplify computations when solving the specific problem. In addition, these formulations enable us to obtain principally different generalizations of equation (1.1.1) for the fields with an arbitrary spin [115, 116].

3. Lie symmetry of the Dirac equation. We adduce the assertions describing the maximal (in Lie sense) invariance groups admitted by the Dirac equation.

Theorem 1.1.1. *The maximal local invariance group of the Dirac equation (1.1.1) is the 14-parameter group* $G_1 = P(1,3) \otimes V(4)$,[1] *where* $P(1,3)$ *is the Poincaré group having the generators*

$$P_\mu = \partial^\mu, \quad J_{\mu\nu} = x_\mu \partial^\nu - x_\nu \partial^\mu + S_{\mu\nu} \tag{1.1.20}$$

and $V(4)$ *is the 4-parameter group of transformations in the space* (ψ^*, ψ) *generated by the operators*

$$\begin{aligned}
Q_0 &= \psi^\alpha \partial_{\psi^\alpha} + \psi^{*\alpha} \partial_{\psi^{*\alpha}}, \\
Q_1 &= i\psi^\alpha \partial_{\psi^\alpha} - i\psi^{*\alpha} \partial_{\psi^{*\alpha}}, \\
Q_2 &= \{\gamma_2 \psi^*\}^\alpha \partial_{\psi^\alpha} - \{\gamma_2 \psi\}^\alpha \partial_{\psi^{*\alpha}}, \\
Q_3 &= \{i\gamma_2 \psi^*\}^\alpha \partial_{\psi^\alpha} + \{i\gamma_2 \psi\}^\alpha \partial_{\psi^{*\alpha}}.
\end{aligned} \tag{1.1.21}$$

In formulae (1.1.20), (1.1.21) $\{\Psi\}^\alpha$ is the α-th component of the function Ψ and

$$\begin{aligned}
S_{\mu\nu} &= \frac{1}{4}[\gamma_\mu, \gamma_\nu] = \frac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \\
\partial_{\psi^\alpha} &= \partial / \partial \psi^\alpha, \quad \partial_{\psi^{*\alpha}} = \partial / \partial \psi^{*\alpha}.
\end{aligned}$$

Theorem 1.1.2. *The maximal local invariance group of the massless Dirac equation (1.1.17) is the 23-parameter group* $G_2 = C(1,3) \otimes V(8)$,[2] *where* $C(1,3)$ *is the conformal group having the generators*

$$\begin{aligned}
P_\mu &= \partial^\mu, \quad J_{\mu\nu} = x_\mu \partial^\nu - x_\nu \partial^\mu + S_{\mu\nu}, \\
D &= x_\mu \partial_\mu + 3/2, \\
K_\mu &= 2x_\mu (x_\nu \partial_\nu + 3/2) - x \cdot x \partial^\mu + 2S_{\mu\nu} x^\nu
\end{aligned} \tag{1.1.22}$$

and $V(8)$ *is the 8-parameter group of transformations in the space* (ψ^*, ψ)

[1]Since equation (1.1.1) is linear, it admits an infinite-parameter group $\psi' = \psi + \theta\Psi(x)$, where θ is a group parameter and Ψ is an arbitrary solution of system of PDEs (1.1.1). Such a symmetry gives no essential information about the structure of solutions of the equation under study and therefore is neglected.

[2]See the previous footnote.

generated by the operators (1.1.21) and

$$
\begin{aligned}
Q_4 &= \{\gamma_4\psi\}^\alpha \partial_{\psi^\alpha} - \{\gamma_4\psi^*\}^\alpha \partial_{\psi^{*\alpha}},\\
Q_5 &= \{i\gamma_4\psi\}^\alpha \partial_{\psi^\alpha} + \{i\gamma_4\psi^*\}^\alpha \partial_{\psi^{*\alpha}},\\
Q_6 &= \{\gamma_2\gamma_4\psi^*\}^\alpha \partial_{\psi^\alpha} + \{\gamma_2\gamma_4\psi\}^\alpha \partial_{\psi^{*\alpha}},\\
Q_7 &= \{i\gamma_2\gamma_4\psi^*\}^\alpha \partial_{\psi^\alpha} - \{i\gamma_2\gamma_4\psi\}^\alpha \partial_{\psi^{*\alpha}}.
\end{aligned}
\tag{1.1.23}
$$

The fact that the groups $G_1,\ G_2$ are the maximal invariance groups admitted by equation (1.1.1) is established by rather cumbersome computations with the help of the Lie method [63, 188].

Straightforward verification shows that operators $P_\mu,\ J_{\mu\nu},\ D$ satisfy the following commutation relations:

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha,\\
&[P_\mu, D] = P_\mu, \quad [J_{\mu\nu}, D] = 0,\\
&[J_{\mu\nu}, J_{\alpha\beta}] = g_{\mu\beta}J_{\nu\alpha} + g_{\nu\alpha}J_{\mu\beta} - g_{\mu\alpha}J_{\nu\beta} - g_{\nu\beta}J_{\mu\alpha}.
\end{aligned}
$$

Consequently, the operators $P_\mu,\ J_{\mu\nu},\ D$ form a basis of the 11-dimensional Lie algebra which is called the extended Poincaré algebra $A\widetilde{P}(1,3)$. The corresponding Lie group is called the extended Poincaré group $\widetilde{P}(1,3)$.

Let us adduce explicit forms of transformation groups generated by operators (1.1.20)–(1.1.23) (corresponding formulae are obtained by solving the Lie equations (0.6)).

1) the group of translations ($X = \theta^\mu P_\mu$)

$$
x'_\mu = x_\mu + \theta_\mu, \quad \psi'(x') = \psi(x); \tag{1.1.24}
$$

2) the Lorentz group $O(1,3)$
a) the rotation group $O(3)$ ($X = \frac{1}{2}\varepsilon_{abc}\theta_a J_{bc}$)

$$
\begin{aligned}
&x'_0 = x_0,\\
&x'_a = x_a\cos\theta - \theta^{-1}\sin\theta\,\varepsilon_{abc}\theta_b x_c + \theta^{-2}\theta_a(1-\cos\theta)\theta_b x_b,\\
&\psi'(x') = \exp\Big\{-\frac{1}{2}\varepsilon_{abc}\theta_a S_{bc}\Big\}\psi(x);
\end{aligned}
\tag{1.1.25}
$$

b) the Lorentz transformations ($X = J_{0a}$)

$$
\begin{aligned}
x'_0 &= x_0\cosh\theta_0 + x_a\sinh\theta_0,\\
x'_a &= x_a\cosh\theta_0 + x_0\sinh\theta_0,
\end{aligned}
$$

$$x'_b = x_b, \quad b \neq a, \qquad \psi'(x') = \exp\left\{\frac{\theta_0}{2}\gamma_0\gamma_a\right\}\psi(x); \tag{1.1.26}$$

3) the group of scale transformations ($X = D$)

$$x'_\mu = e^{\theta_0}x_\mu, \quad \psi'(x') = e^{-k\theta_0}\psi(x), \quad \text{with } k = 3/2; \tag{1.1.27}$$

4) the group of special conformal transformations ($X = \theta_\mu K^\mu$)

$$\begin{aligned} x'_\mu &= (x_\mu - \theta_\mu x \cdot x)\sigma^{-1}(x), \\ \psi'(x') &= \sigma(x)(1 - \gamma \cdot \theta\gamma \cdot x)\psi(x); \end{aligned} \tag{1.1.28}$$

5) the group $V(8)$

$$\begin{aligned}
X = Q_0: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = e^{\theta_0}\psi(x), \quad \psi^{*\prime}(x') = e^{\theta_0}\psi^*(x); \\
X = Q_1: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = e^{i\theta_0}\psi(x), \quad \psi^{*\prime} = e^{-i\theta_0}\psi^*(x); \\
X = Q_2: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = \psi(x)\cosh\theta_0 + \gamma_2\psi^*(x)\sinh\theta_0, \\
& \psi^{*\prime}(x') = \psi^*(x)\cosh\theta_0 - \gamma_2\psi(x)\sinh\theta_0; \\
X = Q_3: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = \psi(x)\cosh\theta_0 + i\gamma_2\psi^*(x)\sinh\theta_0, \\
& \psi^{*\prime}(x') = \psi^*(x)\cosh\theta_0 + i\gamma_2\psi(x)\sinh\theta_0;
\end{aligned} \tag{1.1.29}$$

$$\begin{aligned}
X = Q_4: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = \exp\{\theta_0\gamma_4\}\psi(x), \quad \psi^{*\prime}(x') = \exp\{-\theta_0\gamma_4\}\psi^*(x); \\
X = Q_5: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = \exp\{i\theta_0\gamma_4\}\psi(x), \quad \psi^{*\prime}(x') = \exp\{i\theta_0\gamma_4\}\psi^*(x); \\
X = Q_6: \quad & x'_\mu = x_\mu, \\
& \psi'(x') = \psi(x)\cos\theta_0 + \gamma_2\gamma_4\psi^*(x)\sin\theta_0, \\
& \psi^{*\prime}(x') = \psi^*(x)\cos\theta_0 + \gamma_2\gamma_4\psi(x)\sin\theta_0; \\
X = Q_7: \quad & x'_\mu = x_\mu,
\end{aligned} \tag{1.1.30}$$

$$\psi'(x') = \psi(x)\cos\theta_0 + i\gamma_2\gamma_4\psi^*(x)\sin\theta_0,$$
$$\psi^{*\prime}(x') = \psi^*(x)\cos\theta_0 - i\gamma_2\gamma_4\psi(x)\sin\theta_0.$$

In the above formulae $\theta_\mu \in \mathbb{R}^1$, $\mu = 0, \dots, 3$ are group parameters, $\theta = (\theta_a\theta_a)^{1/2}$, $\sigma(x) = 1 - 2\theta \cdot x + (\theta \cdot \theta)(x \cdot x)$, by the symbol X we designate a generator of the corresponding group.

The direct verification shows that the Dirac equation is invariant under the Lie transformation groups (1.1.24)–(1.1.30). For example, if we make in PDE (1.1.1) the change of variables (1.1.25), then the identity holds

$$(i\gamma_\mu\partial'_\mu - m)\psi'(x') = \exp\left\{\frac{1}{2}\varepsilon_{abc}\theta_a S_{bc}\right\}(i\gamma_\mu\partial_\mu - m)\psi(x),$$

whence it follows that the set of solutions of equation (1.1.1) is invariant with respect to the action of the group (1.1.25).

In addition, the Dirac equation admits discrete transformation groups which cannot be obtained with the help of the Lie method. We adduce the most important discrete symmetries of equation (1.1.1).

1) the spatial inversion

$$\begin{aligned} &x'_0 = x_0, \quad x'_a = -x_a, \\ &\psi'(x') = \gamma_0\psi(x), \quad \psi^{*\prime}(x') = \gamma_0\psi^*(x); \end{aligned} \tag{1.1.31}$$

2) the time reversal

$$\begin{aligned} &x'_0 = -x_0, \quad x'_a = x_a, \\ &\psi'(x') = \gamma_1\gamma_3\psi^*(x), \quad \psi^{*\prime}(x') = \gamma_1\gamma_3\psi(x); \end{aligned} \tag{1.1.32}$$

3) the charge conjugation

$$\begin{aligned} &x'_\mu = x_\mu, \\ &\psi'(x') = i\gamma_2\psi^*(x), \quad \psi^{*\prime}(x') = i\gamma_2\psi(x). \end{aligned}$$

Transformation groups (1.1.31), (1.1.32), (1.1.25), (1.1.26) form the full Lorentz group (for more detail, see [118, 174]).

4. Non-Lie symmetry of the Dirac equation. In the previous subsection we adduced theorems describing maximal local invariance groups of equations (1.1.1), (1.1.17). Such a symmetry can be defined as invariance with respect to a Lie algebra having basis elements of the form

$$X = \xi_\mu(x, \psi, \psi^*)\partial_\mu + \eta^\alpha(x, \psi, \psi^*)\partial_{\psi^\alpha} + \eta^{*\alpha}(x, \psi, \psi^*)\partial_{\psi^{*\alpha}}, \tag{1.1.33}$$

where ξ_μ, η^α, $\eta^{*\alpha}$ are some scalar smooth functions.

As pointed out in the introduction, the above symmetry does not exhaust all symmetry properties of the Dirac equation because there exist linear differential and integro-differential symmetry operators which cannot be represented in the form (1.1.33) and, consequently, correspond to a non-Lie symmetry of the Dirac equation.

Let $\mathcal{M}_1$ be a class of complex linear first-order differential operators with variable matrix coefficients acting on the space of four-component functions, i.e.,

$$\mathcal{M}_1 = \Big\{Q = A_\mu \partial_\mu + B\Big\},$$

where $A_\mu(x)$, $B(x)$ are complex (4×4)-matrices. Evidently, the class $\mathcal{M}_1$ contains all Lie symmetry operators which can be obtained with the help of formula (0.12).

Following [118] we adduce assertions describing all symmetry operators of the Dirac equation belonging to the class $\mathcal{M}_1$.

Theorem 1.1.3 [118, 196, 255]. *Equation (1.1.1) has 26 linearly-independent symmetry operators belonging to the class $\mathcal{M}_1$. The list of these operators is exhausted by the generators of the Poincaré group (1.1.20) and by the following operators:*

$$\begin{aligned}
&I, \quad B = \gamma_4(ix_\mu\partial_\mu + 3i/2 - m\gamma\cdot x),\\
&\omega_{\mu\nu} = (i/2)(\gamma_\mu\partial^\nu - \gamma_\nu\partial^\mu) + mS_{\mu\nu},\\
&\rho_\mu = (1/2)\gamma_4(i\partial^\mu - m\gamma_\mu),\\
&R_\mu = x^\nu\omega_{\mu\nu} + \omega_{\mu\nu}x^\nu,
\end{aligned} \tag{1.1.34}$$

where I is the unit (4×4)-matrix.

Note 1.1.1. Set of operators (1.1.20), (1.1.34) is not closed with respect to the algebraic operation

$$Q_1, Q_2 \to Q_3 = [Q_1,\, Q_2].$$

Consequently, it does not form a Lie algebra. Nevertheless, there exist such subsets of the above set which are Lie algebras. An important example is provided by the operators P_μ, $J_{\mu\nu}$ satisfying the commutation relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu,\\
&[J_{\mu\nu}, J_{\alpha\beta}] = g_{\mu\beta}J_{\nu\alpha} + g_{\nu\alpha}J_{\mu\beta} - g_{\mu\alpha}J_{\nu\beta} - g_{\nu\beta}J_{\mu\alpha}.
\end{aligned} \tag{1.1.35}$$

This algebra is called the Lie algebra of the Poincaré group (or Poincaré algebra) and is designated by the symbol $AP(1,3)$.

Another interesting example is the eight-dimensional Lie algebra

$$\begin{aligned}&\Sigma_{\mu\nu} = mS_{\mu\nu} + (i/2)(1 - i\gamma_4)(\gamma_\mu\partial^\nu - \gamma_\nu\partial^\mu),\\ &\Sigma_0 = I, \quad \Sigma_1 = m\gamma_4 + (1 - i\gamma_4)\gamma_\mu\partial_\mu\end{aligned}$$

obtained in [115, 116].

Note 1.1.2. As the direct check shows the relations

$$\begin{aligned}&B = -\varepsilon_{\mu\nu\alpha\beta}J^{\mu\nu}J^{\alpha\beta}, \quad [P_\mu, B] = 2\rho_\mu,\\ &[\rho_\mu, B] = (1/2)(P_\mu + mR_\mu),\\ &[P_\mu, R_\nu] = 2\omega_{\mu\nu}\end{aligned}$$

hold true. Hence it follows that all symmetry operators of the Dirac equation $Q \in \mathcal{M}_1$ belong to the enveloping algebra of the Poincaré algebra (i.e., to the algebra whose basis elements are polynomials in P_μ, $J_{\mu\nu}$ with constant coefficients). Furthermore, any linear N-th order partial differential operator with matrix coefficients which is a symmetry operator of the Dirac equation (1.1.1) under $m \neq 0$ belongs to the enveloping algebra of the Poincaré algebra [118].

Theorem 1.1.4 [118, 230]. *The massless Dirac equation (1.1.17) has 52 linearly-independent symmetry operators belonging to the class $\mathcal{M}_1$. A basis of the linear vector space of such operators can be chosen as follows*

$$\begin{aligned}&P_\mu, \quad J_{\mu\nu}, \quad K_\mu, \quad D, \quad I, \quad \widetilde{P}_\mu = i\gamma_4 P_\mu,\\ &\widetilde{J}_{\mu\nu} = i\gamma_4 J_{\mu\nu}, \quad \widetilde{D} = i\gamma_4 D, \quad \widetilde{K}_\mu = i\gamma_4 K_\mu,\\ &F = i\gamma_4, \quad R_\mu = (D - 1/2)\gamma_\mu - \gamma\cdot x P_\mu,\\ &\widetilde{R}_\mu = i\gamma_4 R_\mu, \quad \omega_{\mu\nu} = \gamma_\mu P_\nu - \gamma_\nu P_\mu,\\ &Q_{\mu\nu} = i([R_\mu, K_\nu] - [R_\nu, K_\mu]).\end{aligned} \tag{1.1.36}$$

Note 1.1.3. Operators R_μ, $\widetilde{R}_\mu$, $\omega_{\mu\nu}$, $Q_{\mu\nu}$ are not contained in the enveloping algebra of the local invariance algebra of equation (1.1.17). Consequently, they are essentially new.

Until now when analyzing non-Lie symmetry of the Dirac equation we considered only linear transformations of the set of its solutions. To investigate symmetry of equation (1.1.1) in the class of operators generating both linear

and anti-linear transformations (i.e., transformations of the form

$$\begin{aligned} \psi' &= L_1\psi + L_2\psi^*, \\ \psi^{*\prime} &= L_1^*\psi^* + L_2^*\psi, \end{aligned}$$

where L_1, L_2 are some linear differential operators) we turn to the eight-component form of the Dirac equation (1.1.13).

Let $\mathcal{M}_2$ be a class of complex first-order linear differential operators with matrix coefficients

$$X = A_\mu(x)\partial_\mu + B(x)$$

acting on the space of eight-component functions $\Psi = \Psi(x)$.

Theorem 1.1.5 [118]. *The general form of a symmetry operator for equation (1.1.13) belonging to the class $\mathcal{M}_2$ is given by the formula*

$$Q = \begin{pmatrix} Q_0 & -\gamma_2 Q_1^*\gamma_2 \\ Q_1 & -\gamma_2 Q_0^*\gamma_2 \end{pmatrix},$$

where $Q_0,\ Q_1$ are arbitrary linear combinations of the generators of the Poincaré group and of operators (1.1.34) with complex coefficients.

Theorem 1.1.6 [118]. *The general form of a symmetry operator for equation (1.1.13) with $m = 0$ belonging to the class $\mathcal{M}_2$ is given by the formula*

$$Q = \begin{pmatrix} Q_0 & -\gamma_2 Q_1^*\gamma_2 \\ Q_1 & -\gamma_2 Q_0^*\gamma_2 \end{pmatrix},$$

where $Q_0,\ Q_1$ are arbitrary linear combinations of the operators (1.1.36) with complex coefficients.

A detailed account of symmetry properties of the linear Dirac equation in the class of high-order differential and integro-differential operators is given in the monographs [115, 116, 118, 119].

It is well-known that the maximal (in Lie sense) invariance group of the Weyl equation

$$(i\partial_0 + i\sigma_a\partial_a)\varphi(x) = 0 \tag{1.1.37}$$

is the conformal group $C(1,3)$ supplemented by the two-dimensional transformation group

$$\begin{aligned} &x'_\mu = x_\mu, \quad \mu = 0, \dots, 3, \\ &\varphi'(x') = e^{\theta_1 + i\theta_2}\varphi(x), \end{aligned}$$

where $\{\theta_1, \theta_2\} \subset \mathbb{R}^1$.

The class $\mathcal{M}_1$ has no additional symmetry operators. The class $\mathcal{M}_2$ contains 52 symmetry operators for the Weyl equation [230].

5. Absolute time for the Dirac equation. All fundamental equations of quantum field theory (Maxwell, Dirac, Klein-Gordon-Fock, d'Alembert etc.) are invariant with respect to the Lorentz transformations. With these transformations time changes after transfer from one inertial coordinate system to another. In other words, the principal motion equations of the quantum field theory are invariant with respect to the Lorentz group $O(1,3) \in P(1,3)$.

A question arises whether there exist invariance algebras admitted by the Maxwell, Dirac and Klein-Gordon-Fock equations which generate transformations for the time variable $x_0 \equiv t$ and coordinates $\vec{x} \equiv (x_1, x_2, x_3)$ different from the Lorentz and Galilei transformations. A positive answer to this question was given in the papers [83]–[86].

Theorem 1.1.7 [83]–[86]. *The Dirac equation (1.1.1) is invariant under the Poincaré algebra having the following basis elements:*

$$
\begin{aligned}
&P_0^{(1)} \equiv H = -\gamma_0\gamma_a\partial_a - im\gamma_0, \quad P_a^{(1)} = -\partial_a,\\
&J_{0a}^{(1)} = -x_0\partial_a - \tfrac{1}{2}(x_a H + H x_a),\\
&J_{ab}^{(1)} = x_b\partial_a - x_a\partial_b + \tfrac{1}{2}\gamma_a\gamma_b,
\end{aligned}
\tag{1.1.38}
$$

where $a, b = 1, 2, 3, \ a < b$.

Proof is carried out by direct check.

Note 1.1.4. The operators $J_{0a}^{(1)}$ generate *non-Lorentz* transformations of the time variable $x_0 = t$ and coordinates x_a. Time does not change

$$
t \to t' = \exp\{v_a J_{0a}^{(1)}\} t \exp\{-v_a J_{0a}^{(1)}\} \equiv t
\tag{1.1.39}
$$

and the coordinates transform as follows:

$$
\begin{aligned}
x_a \to x'_a \quad &= \quad \exp\{v_b J_{0b}^{(1)}\} x_a \exp\{-v_b J_{0b}^{(1)}\}\\
&\neq \quad \underbrace{\exp\{v_b J_{0b}\} x_a \exp\{-v_b J_{0b}\}}_{\text{Lorentz transformations}}.
\end{aligned}
\tag{1.1.40}
$$

Here v_b are parameters which are interpreted as components of the velocity of a moving inertial reference frame with respect to a fixed one.

Note 1.1.5. It follows from Theorem 1.1.7 that on the set of solutions of the Dirac equation two inequivalent representations of the Poincaré algebra are

realized. Operators $P_0^{(1)}, J_{0a}^{(1)}$ from (1.1.38) generate nonlocal transformations of coordinates x_a leaving the time variable $x_0 = t$ invariant. Let us emphasize that transformations (1.1.40) are different from the standard Galilei and Lorentz transformations.

As the relations

$$\begin{aligned}
&\left(P_0^{(1)}\right)^2 - \left(P_a^{(1)}\right)\left(P_a^{(1)}\right) = -m^2, \\
&[P_0^{(1)},\, J_{0a}^{(1)}] = P_a^{(1)}, \\
&[P_b^{(1)},\, J_{0a}^{(1)}] = -g_{ab}P_0^{(1)}
\end{aligned}$$

hold, the energy $P_0^{(1)}$ and momentum $P_a^{(1)}$ operators transform according to the standard Lorentz law. But for the time variable $x_0 = t$ and coordinates x_a this is not the case and the interval $s^2 = x_0^2 - x_a x_a$ is not invariant with respect to the transformations (1.1.39), (1.1.40).

Thus, the Dirac equation as well as the Maxwell and the Klein-Gordon-Fock equations [83]–[86] have dual symmetry (Lorentz and non-Lorentz).

The dual symmetry of the Dirac equation is a consequence of the fact that the spectrum of the operator H has a lacuna in the interval $(-m, m)$ and the spectrum of the operator $P_0^{(1)}$ is continuous on the real axis [83]–[86].

In conclusion we briefly consider symmetry properties of the equation

$$(1 - i\gamma_4)\gamma_\mu \partial_\mu \psi = 0, \tag{1.1.41}$$

which is obtained from the massless Dirac equation (1.1.17) by multiplying it by the singular matrix $1 - i\gamma_4$. This equation is distinguished by the fact that two inequivalent representations of the conformal group $C(1,3)$ are realized on the set of its solutions. The first one is given by formulae (1.1.24)–(1.1.28). In addition, equation (1.1.41) admits the group $C(1,3)$ with generators $P_\mu,\ J_{\mu\nu}$ of the form (1.1.20) and

$$\begin{aligned}
&D = -x_\mu \partial_\mu - 3/2 + \lambda_1(i\gamma_4 - 1), \\
&K_\mu = 2x_\mu D - x \cdot x \partial^\mu - 2S_{\mu\nu}x^\nu + \lambda_2(i\gamma_4 - 1)\gamma_\mu,
\end{aligned} \tag{1.1.42}$$

where $\lambda_1,\ \lambda_2$ are non-zero constants.

From [221] it follows that formulae (1.1.42) determine the most general form of generators of groups of scale and special conformal transformations from the group $C(1,3)$ if the generators of the group $P(1,3)$ are given in covariant form (1.1.20).

It will be shown in Section 2.2 that the representation (1.1.42) plays an important role when constructing conformally-invariant solutions of spinor equations.

1.2. Nonlinear spinor equations

This section is devoted to symmetry analysis of quasi-linear systems of PDEs for the spinor field of the form

$$i\gamma_\mu \partial_\mu \psi - F(\bar{\psi}, \psi) = 0, \tag{1.2.1}$$

where $F = (F^0, F^1, F^2, F^3)^T,\ \ F^\mu \in C^1(\mathbb{C}^8, \mathbb{C}^1)$.

It is clear that an arbitrary equation of the type (1.2.1) cannot be taken as a true nonlinear generalization of the Dirac equation. A natural restriction on the choice of functions F^μ is the condition of invariance under the Poincaré group. This condition provides independence of the choice of inertial reference frame for physical processes described by equation (1.2.1) (i.e., nonlinear PDE (1.2.1) has to satisfy the Lorentz–Poincaré–Einstein relativity principle). Mathematical expression of the above principle is a condition of invariance under the group $P(1,3)$ with generators (1.1.20). In addition, it is of interest to select subclasses of Poincaré-invariant equations of the form (1.2.1) admitting wider symmetry groups – the extended Poincaré group and the conformal group.

Theorem 1.2.1 [152, 155]. *System of nonlinear PDEs (1.2.1) is invariant under the Poincaré group $P(1,3)$ iff*

$$F(\bar{\psi}, \psi) = \{f_1(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi) + f_2(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi)\gamma_4\}\psi, \tag{1.2.2}$$

where $\{f_1,\ f_2\} \subset C^1(\mathbb{R}^2, \mathbb{C}^1)$ are arbitrary functions.

Proof. Without loss of generality equation (1.2.1) can be rewritten in the following form:

$$\{i\gamma_\mu \partial_\mu + \Phi(\bar{\psi}, \psi)\}\psi = 0, \tag{1.2.3}$$

where $\Phi(\bar{\psi}, \psi)$ is a (4×4)-matrix.

It is evident that equation (1.2.3) with an arbitrary matrix function Φ is invariant under the 4-parameter group of translations (1.1.24). Consequently, to prove the theorem it is enough to describe all Φ such that PDE (1.2.3) admits the Lorentz transformations (1.1.26), whence due to the commutation

relations of the algebra $AO(1,3)$ it follows that PDE in question is invariant under the Poincaré group.

Acting with the first prolongation of the operator J_{0a} on equation (1.2.3) and passing to the set of its solutions we obtain a system of PDEs for an unknown matrix function $\Phi(\bar\psi,\psi)$

$$Q_{0a}\Phi + (1/2)(\Phi\gamma_0\gamma_a - \gamma_0\gamma_a\Phi) = 0. \tag{1.2.4}$$

Let us expand the matrix Φ in the complete system of the Dirac matrices I, γ_μ, $S_{\mu\nu}$, $\gamma_4\gamma_\mu$, γ_4

$$\begin{aligned}\Phi \;=\; & A(\bar\psi,\psi) + B^\mu(\bar\psi,\psi)\gamma_\mu + C^{\mu\nu}(\bar\psi,\psi)S_{\mu\nu} \\ & + D^\mu(\bar\psi,\psi)\gamma_4\gamma_\mu + E(\bar\psi,\psi)\gamma_4.\end{aligned} \tag{1.2.5}$$

Substituting expression (1.2.5) into (1.2.4) and taking into account the identities

$$\begin{aligned}&[\gamma_4,\gamma_0\gamma_a] = 0, \quad [\gamma_\mu,\gamma_0\gamma_a] = 2(g_{\mu 0}\gamma_a - g_{\mu a}\gamma_0),\\ &[\gamma_\mu\gamma_\nu,\gamma_0\gamma_a] = 2(g_{\mu 0}\gamma_a\gamma_\nu - g_{\mu a}\gamma_0\gamma_\nu + g_{\nu 0}\gamma_\mu\gamma_a - g_{\nu a}\gamma_\mu\gamma_0),\end{aligned}$$

where $g_{\mu\nu}$ is the metric tensor of the Minkowski space $R(1,3)$, with a subsequent equating to zero of coefficients of linearly independent matrices I, γ_μ, $\ldots$, γ_4 one gets an over-determined system of PDEs for functions A, $B^\mu, \ldots, E$

$$Q_{0a}A = 0, \quad Q_{0a}E = 0, \tag{1.2.6}$$

$$Q_{0a}B_\mu + B^\alpha(g_{\alpha 0}g_{\mu a} - g_{\alpha a}g_{\mu 0}) = 0, \tag{1.2.7}$$

$$Q_{0a}D_\mu + D^\alpha(g_{\alpha 0}g_{\mu a} - g_{\alpha a}\gamma_{\mu 0}) = 0, \tag{1.2.8}$$

$$\begin{aligned}&Q_{0a}C^{\mu\nu} + (1/2)C^{\alpha\beta}(g_{\alpha a}\delta^{\mu\nu}_{\beta 0} + g_{\beta 0}\delta^{\mu\nu}_{\alpha a}\\ &\quad - g_{\alpha 0}\delta^{\mu\nu}_{\beta a} - g_{\beta a}\delta^{\mu\nu}_{\alpha 0}) = 0.\end{aligned} \tag{1.2.9}$$

In formulae (1.2.6)–(1.2.9) we use the following notations:

$$\begin{aligned}&Q_{\mu\nu} = (1/2)\{\gamma_\mu\gamma_\nu\psi\}^\alpha\partial_{\psi^\alpha} - (1/2)\{\bar\psi\gamma_\mu\gamma_\nu\}^\alpha\partial_{\bar\psi^\alpha}, \quad \mu < \nu,\\ &\delta^{\mu\nu}_{\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}, \quad a = 1,2,3, \quad \mu,\nu,\alpha,\beta = 0,1,2,3.\end{aligned}$$

Since $[Q_{0a}, Q_{0b}] = Q_{ab}$, functions $A(\bar\psi,\psi)$, $B(\bar\psi,\psi)$ satisfy the system of PDEs

$$Q_{\mu\nu}f(\bar\psi,\psi) = 0, \quad \mu < \nu. \tag{1.2.10}$$

According to [61], the general solution of this system is an arbitrary smooth function of a complete set of its first integrals ω.

If we denote by r the rank of the (6×8)-matrix of coefficients of the operators $Q_{\mu\nu}$

$$\begin{bmatrix} -\frac{i}{2}\psi^3 & -\frac{i}{2}\psi^2 & -\frac{i}{2}\psi^1 & -\frac{i}{2}\psi^0 & \frac{i}{2}\bar{\psi}^3 & \frac{i}{2}\bar{\psi}^2 & \frac{i}{2}\bar{\psi}^1 & \frac{i}{2}\bar{\psi}^0 \\ -\frac{1}{2}\psi^3 & \frac{1}{2}\psi^2 & -\frac{1}{2}\psi^1 & \frac{1}{2}\psi^0 & -\frac{1}{2}\bar{\psi}^3 & \frac{1}{2}\bar{\psi}^2 & -\frac{1}{2}\bar{\psi}^1 & \frac{1}{2}\bar{\psi}^0 \\ -\frac{i}{2}\psi^2 & \frac{i}{2}\psi^3 & -\frac{i}{2}\psi^0 & \frac{i}{2}\psi^1 & \frac{i}{2}\bar{\psi}^2 & -\frac{i}{2}\bar{\psi}^3 & \frac{i}{2}\bar{\psi}^0 & -\frac{i}{2}\bar{\psi}^1 \\ -\frac{1}{2}\psi^0 & \frac{1}{2}\psi^1 & -\frac{1}{2}\psi^2 & \frac{1}{2}\psi^3 & \frac{1}{2}\bar{\psi}^0 & -\frac{1}{2}\bar{\psi}^1 & \frac{1}{2}\bar{\psi}^2 & -\frac{1}{2}\bar{\psi}^3 \\ -\frac{1}{2}\psi^1 & -\frac{1}{2}\psi^0 & -\frac{1}{2}\psi^3 & -\frac{1}{2}\psi^2 & \frac{1}{2}\bar{\psi}^1 & \frac{1}{2}\bar{\psi}^0 & \frac{1}{2}\bar{\psi}^3 & \frac{1}{2}\bar{\psi}^2 \\ \frac{i}{2}\psi^1 & -\frac{i}{2}\psi^0 & \frac{i}{2}\psi^3 & -\frac{i}{2}\psi^2 & \frac{i}{2}\bar{\psi}^1 & -\frac{i}{2}\bar{\psi}^0 & \frac{i}{2}\bar{\psi}^3 & -\frac{i}{2}\bar{\psi}^2 \end{bmatrix}$$

(the representation of γ-matrices is given by formulae (1.1.5)), then a maximal set of functionally-independent first integrals of system (1.2.10) consists of $8-r$ integrals [61]. In the case considered $r = 6$, whence it follows that the general solution is represented as an arbitrary smooth function of two functionally-independent first integrals. As a rule, they are chosen in the form $\bar{\psi}\psi$, $\bar{\psi}\gamma_4\psi$. Thus, the general solution of system (1.2.6) is given by the formulae

$$A = \widetilde{A}(\bar{\psi}\psi,\ \bar{\psi}\gamma_4\psi), \quad E = \widetilde{E}(\bar{\psi}\psi,\ \bar{\psi}\gamma_4\psi), \tag{1.2.11}$$

where $\{\widetilde{A}, \widetilde{E}\} \subset C^1(\mathbb{R}^2, \mathbb{C}^1)$ are arbitrary functions.

We expand the four-component function with components B_μ in the system of four linearly independent vectors $e_1,\ e_2,\ e_3,\ e_4$ having the components $\bar{\psi}\gamma_\mu\psi,\ \bar{\psi}\gamma_4\gamma_\mu\psi,\ \psi^T\gamma_0\gamma_2\gamma_\mu\psi,\ \psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\psi$

$$\begin{aligned} B_\mu \ = \ & R_1(\bar{\psi},\psi)\bar{\psi}\gamma_\mu\psi + R_2(\bar{\psi},\psi)\bar{\psi}\gamma_4\gamma_\mu\psi \\ & + R_3(\bar{\psi},\psi)\psi^T\gamma_0\gamma_2\gamma_\mu\psi + R_4(\bar{\psi},\psi)\psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\psi. \end{aligned}$$

Let us prove that the functions $B_\mu = B_\mu(\bar{\psi},\psi),\ \mu = 0,\ldots,3$ satisfy system of PDEs (1.2.7) iff the conditions

$$R_i = \widetilde{B}_i(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi), \quad \widetilde{B}_i \in C^1(\mathbb{R}^2,\ \mathbb{C}^1), \quad i = 1,\ldots,4$$

hold.

Indeed, if we designate by $V_\mu(\bar\psi,\psi)$ the components of one of the vectors e_i, then V_μ satisfy the equalities of the form

$$Q_{0a}V_0 = V_a, \quad a = 1,2,3 \tag{1.2.12}$$

(the above fact is established by straightforward computation). Consequently, we have

$$\begin{aligned} Q_{0a}B_0 \;=\;& (Q_{0a}R_1)\bar\psi\gamma_0\psi + (Q_{0a}R_2)\bar\psi\gamma_4\gamma_0\psi + (Q_{0a}R_3) \\ &\times\psi^T\gamma_0\gamma_2\gamma_0\psi + (Q_{0a}R_4)\psi^T\gamma_0\gamma_2\gamma_4\gamma_0\psi + R_1\bar\psi\gamma_a\psi \\ &+R_2\bar\psi\gamma_4\gamma_a\psi + R_3\psi^T\gamma_0\gamma_2\gamma_a\psi + R_4\psi^T\gamma_0\gamma_2\gamma_4\gamma_a\psi. \end{aligned} \tag{1.2.13}$$

Setting $\mu = 0$ in (1.2.7) we find

$$Q_{0a}B_0 = B_a. \tag{1.2.14}$$

Comparing (1.2.13) and (1.2.14) yields the following equality:

$$\begin{aligned} &(Q_{0a}R_1)\bar\psi\gamma_0\psi + (Q_{0a}R_2)\bar\psi\gamma_4\gamma_0\psi + (Q_{0a}R_3)\psi^T\gamma_0\gamma_2\gamma_0\psi \\ &\quad+(Q_{0a}R_4)\psi^T\gamma_0\gamma_2\gamma_4\gamma_0\psi = 0. \end{aligned} \tag{1.2.15}$$

In the same way we obtain equalities of the form

$$\begin{aligned} &(Q_{0a}R_1)\bar\psi\gamma_b\psi + (Q_{0a}R_2)\bar\psi\gamma_4\gamma_b\psi + Q_{0a}R_3)\psi^T\gamma_0\gamma_2\gamma_b\psi \\ &\quad+(Q_{0a}R_4)\psi^T\gamma_0\gamma_2\gamma_4\gamma_b\psi = 0, \end{aligned} \tag{1.2.16}$$

where $a, b = 1,2,3$.

Since four-vectors with components $\bar\psi\gamma_\mu\psi, \dots, \psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\psi$ are linearly-independent, from (1.2.15), (1.2.16) it follows that $Q_{0a}R_i = 0$, $a = 1,2,3$, $i = 1,\dots,4$ or $R_i = \widetilde B_i(\bar\psi\psi, \bar\psi\gamma_4\psi)$, $i = 1,\dots,4$.

Taking into account that system of PDEs (1.2.8) coincides with system (1.2.7) it is easy to write down its general solution

$$\begin{aligned} D_\mu(\bar\psi,\psi) \;=\;& \bar\psi\gamma_\mu\psi\widetilde D_1(\bar\psi\psi,\bar\psi\gamma_4\psi) + \bar\psi\gamma_4\gamma_\mu\psi\widetilde D_2(\bar\psi\psi,\bar\psi\gamma_4\psi) \\ &+\psi^T\gamma_0\gamma_2\gamma_\mu\psi D_3(\bar\psi\psi,\bar\psi\gamma_4\psi) + \psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\psi\widetilde D_4(\bar\psi\psi,\bar\psi\gamma_4\psi), \end{aligned}$$

where $\widetilde D_i \in C^1(\mathbb{R}^1, \mathbb{C}^1)$, $i = 1,\dots,4$ are arbitrary functions.

Integration of equations (1.2.9) is carried out in the same way, as a result we have

$$\begin{aligned} C_{\mu\nu}(\bar\psi,\psi) \;=\;& \bar\psi\gamma_\mu\gamma_\nu\psi\widetilde C_1 + \bar\psi\gamma_4\gamma_\mu\gamma_\nu\psi\widetilde C_2 \\ &+\psi^T\gamma_0\gamma_2\gamma_\mu\gamma_\nu\psi\widetilde C_3 + \psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\gamma_\nu\psi\widetilde C_4, \end{aligned}$$

where $\widetilde{C}_i = \widetilde{C}_i(\bar\psi\psi, \bar\psi\gamma_4\psi),\ i = 1, \ldots, 4$ are arbitrary smooth functions.

Thus, we have proved that equation (1.2.1) is invariant under the Poincaré group iff

$$\begin{aligned} F(\bar\psi, \psi) &= \Phi(\bar\psi, \psi)\psi \\ &\equiv \Big\{ \widetilde{A}I + \widetilde{B}_1\gamma_\mu(\bar\psi\gamma^\mu\psi) + \widetilde{B}_2\gamma_\mu(\bar\psi\gamma_4\gamma^\mu\psi) \\ &+\widetilde{B}_3\gamma_\mu(\psi^T\gamma_0\gamma_2\gamma^\mu\psi) + \widetilde{B}_4\gamma_\mu(\psi^T\gamma_0\gamma_2\gamma_4\gamma^\mu\psi) \\ &+\widetilde{C}_1S_{\mu\nu}(\bar\psi S^{\mu\nu}\psi) + \widetilde{C}_2S_{\mu\nu}(\bar\psi\gamma_4S^{\mu\nu}\psi) \\ &+\widetilde{C}_3S_{\mu\nu}(\psi^T\gamma_0\gamma_2S^{\mu\nu}\psi) + \widetilde{C}_4S_{\mu\nu}(\psi^T\gamma_0\gamma_2\gamma_4S^{\mu\nu}\psi) \\ &+\widetilde{D}_1\gamma_4\gamma_\mu(\bar\psi\gamma^\mu\psi) + \widetilde{D}_2\gamma_4\gamma_\mu(\bar\psi\gamma_4\gamma^\mu\psi) \\ &+\widetilde{D}_3\gamma_4\gamma_\mu(\psi^T\gamma_0\gamma_2\gamma^\mu\psi) + \widetilde{D}_4\gamma_4\gamma_\mu(\psi^T\gamma_0\gamma_2\gamma_4\gamma^\mu\psi) + \widetilde{E}\gamma_4\Big\}\psi. \end{aligned} \tag{1.2.17}$$

Here $\widetilde{A},\ \widetilde{B}_1, \ldots, \widetilde{E}$ are arbitrary smooth functions of $\bar\psi\psi,\ \bar\psi\gamma_4\psi$.

Let us show that formula (1.2.17) without loss of generality can be rewritten in the form (1.2.2). To this end, we need the following identity:

$$(\bar\psi_1\gamma_\mu\psi_2)\gamma^\mu\psi_2 = (\bar\psi_1\psi_2)\psi_2 + (\bar\psi_1\gamma_4\psi_2)\gamma_4\psi_2, \tag{1.2.18}$$

where $\psi_1,\ \psi_2$ are arbitrary four-component functions.

The validity of (1.2.18) is checked by direct computation. Choosing γ-matrices in the representation (1.1.5) we have

$$\begin{aligned} &\gamma_0\psi_2 = (\psi_2^0,\, \psi_2^1,\, -\psi_2^2,\, -\psi_2^3)^T, \\ &\gamma_1\psi_2 = (\psi_2^3,\, \psi_2^2,\, -\psi_2^1,\, -\psi_2^0)^T, \\ &\gamma_2\psi_2 = (i\psi_2^3,\, -i\psi_2^2,\, -i\psi_2^1,\, i\psi_2^0)^T, \\ &\gamma_3\psi_2 = (\psi_2^2,\, -\psi_2^3,\, -\psi_2^0,\, \psi_2^1)^T, \\ &\bar\psi_1\gamma_0\psi_2 = \bar\psi_1^0\psi_2^0 + \bar\psi_1^1\psi_2^1 - \bar\psi_1^2\psi_2^2 - \bar\psi_1^3\psi_2^3, \\ &\bar\psi_1\gamma_1\psi_2 = \bar\psi_1^0\psi_2^3 + \bar\psi_1^1\psi_2^2 - \bar\psi_1^2\psi_2^1 - \bar\psi_1^3\psi_2^0, \\ &\bar\psi_1\gamma_2\psi_2 = i(\bar\psi_1^0\psi_2^3 - \bar\psi_1^1\psi_2^2 - \bar\psi_1^2\psi_2^1 + \bar\psi_1^3\psi_2^0), \\ &\bar\psi_1\gamma_3\psi_2 = \bar\psi_1^0\psi_2^2 - \bar\psi_1^1\psi_2^3 - \bar\psi_1^2\psi_2^0 + \bar\psi_1^3\psi_2^1, \\ &\bar\psi_1\psi_2 = \bar\psi_1^0\psi_2^0 + \bar\psi_1^1\psi_2^1 + \bar\psi_1^2\psi_2^2 + \bar\psi_1^3\psi_2^3, \\ &\bar\psi_1\gamma_4\psi = -(\bar\psi_1^0\psi_2^2 + \bar\psi_1^1\psi_2^3 + \bar\psi_1^2\psi_2^0 + \bar\psi_1^3\psi_2^1), \end{aligned}$$

whence it follows

$$(\bar\psi_1\gamma_\mu\psi_2)\gamma^\mu\psi_2 = (\bar\psi_1^0\psi_2^0 + \bar\psi_1^1\psi_2^1 + \bar\psi_1^2\psi_2^2 + \bar\psi_1^3\psi_2^3)$$

$$\times\begin{pmatrix}\psi_2^0\\ \psi_2^1\\ \psi_2^2\\ \psi_2^3\end{pmatrix}-(\bar\psi_1^0\psi_2^2+\bar\psi_1^1\psi_2^3+\bar\psi_1^2\psi_2^0+\bar\psi_1^3\psi_2^1)\begin{pmatrix}\psi_2^2\\ \psi_2^3\\ \psi_2^0\\ \psi_2^1\end{pmatrix}$$

$$=\{\bar\psi_1\psi_2+(\bar\psi_1\gamma_4\psi_2)\gamma_4\}\psi_2.$$

On making in (1.2.18) the change of variables $\bar\psi_1\to\bar\psi_1\gamma_4$ we arrive at the identity

$$(\bar\psi_1\gamma_4\gamma_\mu\psi_2)\gamma^\mu\psi_2=\{\bar\psi_1\gamma_4\psi_2-(\bar\psi_1\psi_2)\gamma_4\}\psi_2. \tag{1.2.19}$$

Similarly, we obtain from (1.2.18) two other identities

$$(\bar\psi_1\gamma_4\gamma_\mu\psi_2)\gamma_4\gamma^\mu\psi_2=\{(\bar\psi_1\gamma_4\psi_2)\gamma_4+\bar\psi_1\psi_2\}\psi_2, \tag{1.2.20}$$
$$(\bar\psi_1 S_{\mu\nu}\psi_2)S^{\mu\nu}\psi_2=(1/2)\{\bar\psi_1\psi_2-(\bar\psi_1\gamma_4\psi_2)\gamma_4\}\psi_2. \tag{1.2.21}$$

In (1.2.19)–(1.2.21) ψ_1,ψ_2 are arbitrary four-component functions.

Choosing in (1.2.18), (1.2.19)–(1.2.21) functions $\psi_1,\ \psi_2$ in an appropriate way we arrive at the following relations:

$$\begin{aligned}&(\bar\psi\gamma_\mu\psi)\gamma^\mu\psi=\{\bar\psi\psi+(\bar\psi\gamma_4\psi)\gamma_4\}\psi,\\&(\bar\psi\gamma_4\gamma_\mu\psi)\gamma^\mu\psi=\{\bar\psi\gamma_4\psi-(\bar\psi\psi)\gamma_4\}\psi,\\&\ldots\\&(\psi^T\gamma_0\gamma_2\gamma_4\gamma_\mu\psi)\gamma^\mu\psi=\{\psi^T\gamma_0\gamma_2\psi+(\psi^T\gamma_0\gamma_2\gamma_4\psi)\gamma_4\}\psi=0,\end{aligned}$$

whence the existence of such smooth functions $f_1(\bar\psi\psi,\bar\psi\gamma_4\psi)$, $f_2(\bar\psi\psi,\bar\psi\gamma_4\psi)$ that $\Phi(\bar\psi,\psi)\psi=(f_1+f_2\gamma_4)\psi$ follows. The theorem is proved. $\triangleright$

Note 1.2.1. If we choose in (1.2.17) $\widetilde D_2=\lambda=\text{const}$, $\widetilde A=\widetilde B_1=\ldots=\widetilde D_1=\widetilde D_3=\widetilde D_4=\widetilde E=0$, then equation (1.2.1) coincides with the nonlinear spinor equation (0.1) suggested by Heisenberg.

Note 1.2.2. From formulae (1.2.18)–(1.2.21) the well-known Pauli–Fierz identities follow [62, 274, 275]

$$v_\mu v^\mu=s^2+p^2,\quad w_\mu w^\mu=s^2+p^2,\quad \sigma_{\mu\nu}\sigma^{\mu\nu}=(1/2)(s^2-p^2),$$

where

$$\begin{aligned}&s=\bar\psi\psi,\quad p=\bar\psi\gamma_4\psi,\quad v_\mu=\bar\psi\gamma_\mu\psi,\\&w_\mu=\bar\psi\gamma_4\gamma_\mu\psi,\quad \sigma_{\mu\nu}=\bar\psi S_{\mu\nu}\psi,\quad \mu,\nu=0,\ldots,3.\end{aligned}$$

Further we will select subclasses of equations of the form (1.2.1) which in addition to the group $P(1,3)$ admit the one-parameter group of scale transformations (1.1.27) with arbitrary non-zero $k \in \mathbb{R}^1$ and the 4-parameter group of special conformal transformations.

Theorem 1.2.2 [152, 155]. *Equation (1.2.1) is invariant under the extended Poincaré group, iff the function $F(\bar{\psi}, \psi)$ has the form (1.2.2), where*

$$f_i = (\bar{\psi}\psi)^{1/2k} \tilde{f}_i\Big(\bar{\psi}\psi(\bar{\psi}\psi\gamma_4\psi)^{-1}\Big), \ i = 1, 2. \tag{1.2.22}$$

Proof. The necessity. Since PDE (1.2.1) is invariant under the group $\widetilde{P}(1,3)$, it admits the group $P(1,3) \subset \widetilde{P}(1,3)$. Applying Theorem 1.2.1 we conclude that it is necessary to describe all functions $f_1(\bar{\psi}\psi,\ \bar{\psi}\gamma_4\psi)$, $f_2(\bar{\psi}\psi,\ \bar{\psi}\gamma_4\psi)$ such that equation (1.2.1) with F of the form (1.2.2) is invariant under the group of transformations (1.1.27). Acting by the first prolongation of the infinitesimal generator of the group (1.1.27)

$$D = x_\mu \partial_\mu - k\psi^\alpha \partial \psi^\alpha - k\bar{\psi}_\alpha \partial_{\bar{\psi}^\alpha}$$

on equation (1.2.1) with F of the form (1.2.2) and passing to the set of its solutions yield determining equations for $f_1,\ f_2$

$$\Big(\omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} - (2k)^{-1}\Big) f_i = 0, \ i = 1, 2, \tag{1.2.23}$$

where $\omega_1 = \bar{\psi}\psi,\ \omega_2 = \bar{\psi}\gamma_4\psi$.

The general solutions of the above equations are given by formulae (1.2.22). The necessity is proved.

The sufficiency. Let us introduce a notation

$$G(\bar{\psi}, \psi) = i\gamma_\mu\partial_\mu\psi - (\tilde{f}_1 + \tilde{f}_2\gamma_4)(\bar{\psi}\psi)^{1/2k}\psi. \tag{1.2.24}$$

The direct computation yields the following identity:

$$G(\bar{\psi}', \psi') = e^{(k+1)\theta} G(\bar{\psi}, \psi), \quad \theta \in \mathbb{R}^1,$$

where ψ' is given by formulae (1.1.27).

In other words, the group of scale transformations leaves the set of solutions of equation $G = 0$ invariant. Hence it follows that equation (1.2.1), where the function $F(\bar{\psi}, \psi)$ is determined by (1.2.2), (1.2.22), admits the extended Poincaré group. Theorem is proved. $\rhd$

Theorem 1.2.3 [152, 155]. *Equation (1.2.1) is invariant under the conformal group $C(1,3)$ iff*

$$F(\bar{\psi},\psi) = (\bar{\psi}\psi)^{1/3}(\tilde{f}_1 + \tilde{f}_2\gamma_4)\psi, \tag{1.2.25}$$

where f_1, f_2 are arbitrary smooth functions of $\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}$.

Proof. The necessity. Since the group $C(1,3)$ contains the extended Poincaré group, the function $F(\bar{\psi},\psi)$ has the form (1.2.2), (1.2.22), the conformal degree k being equal to $3/2$.

The sufficiency is established by direct verification. Making the change of variables (1.1.28) in equation $G = 0$, where G is given by (1.2.24) under $k = 3/2$, we get the identity

$$G(\bar{\psi}', \psi') = \sigma^2(x)(1 - \gamma\cdot\,\theta\gamma\cdot x)G(\bar{\psi},\psi),$$

whence it follows that equation $G = 0$ admits the 4-parameter group of special conformal transformations. The theorem is proved. $\rhd$

Note 1.2.3. If we choose in (1.2.25) $\tilde{f}_1 = \lambda = \text{const}$, $\tilde{f}_2 = 0$, then the conformally-invariant spinor equation suggested by Gürsey [176]

$$\{i\gamma_\mu\partial_\mu - \lambda(\bar{\psi}\psi)^{1/3}\}\psi = 0 \tag{1.2.26}$$

is obtained. In addition, by using formulae (1.2.18)–(1.2.21) it is not difficult to become convinced of that the conformally-invariant spinor equation

$$i\{\gamma_\mu\partial_\mu - \lambda[(\bar{\psi}\gamma_4\gamma_\mu\psi)(\bar{\psi}\gamma_4\gamma^\mu\psi)]^{-1/3}(\bar{\psi}\gamma_4\gamma_\mu\psi)\gamma_4\gamma^\mu\}\psi = 0$$

suggested in [139, 140] is also included into the class of nonlinear PDEs (1.2.1), (1.2.25).

Note 1.2.4. Applying the Lie method we can establish that Poincaré-invariant equations (1.2.1), (1.2.2) admit the three-parameter Pauli-Gürsey group having generators Q_1, Q_2, Q_3 (1.1.21) iff the functions f_1, f_2 are real-valued ones.

It should be noted that there exist nonlinear spinor equations which admit infinite-parameter symmetry groups. As an example, we give the following $P(1,3)$-invariant spinor equation:

$$(\bar{\psi}\gamma_\mu\psi)\partial_\mu\psi = 0 \tag{1.2.27}$$

which is obtained from (1.1.17) by a formal change $\gamma_\mu \to \bar{\psi}\gamma_\mu\psi$. The maximal symmetry group of the above equation is generated by the infinitesimal

operator [155, 160]

$$X = \xi_\mu(x, \bar\psi, \psi)\partial_\mu + \eta^\alpha(x, \bar\psi, \psi)\partial_{\psi\alpha} + \bar\eta^\alpha(x, \bar\psi, \psi)\partial_{\bar\psi^\alpha},$$

where

$$\begin{aligned}
\xi_\mu &= f_\mu(w, \bar\psi, \psi) + \bar\psi\gamma_\mu\psi f(x, \bar\psi, \psi) + \bar\psi\gamma\cdot x\psi \\
&\quad \times(\bar R\gamma_\mu\psi + \bar\psi\gamma_\mu R)\{(\bar\psi\gamma_\nu\psi)(\bar\psi\gamma^\nu\psi)\}^{-1}, \\
\eta^\alpha &= R^\alpha(w, \bar\psi, \psi), \\
w &= \{x_\mu(\bar\psi\gamma_\nu\psi)(\bar\psi\gamma^\nu\psi) - (\bar\psi\gamma_\mu\psi)(\bar\psi\gamma\cdot x\psi)\},
\end{aligned}$$

$f,\ f_\mu,\ R^\alpha$ are arbitrary smooth functions and $\mu, \nu, \alpha = 0, \ldots, 3$.

1.3. Systems of nonlinear second-order equations for the spinor field

As a rule, the spinor field is described by the first-order system of PDEs. Such description is considered to be the most adequate to the nature of the spinor field. But there exists another approach based on the second-order equations [89, 91, 241, 242].

Each component of the Dirac spinor satisfies the second-order wave equation (see Section 1.1)

$$(\partial_\mu\partial^\mu + m^2)\psi(x) = 0. \tag{1.3.1}$$

The above equations form a system of splitting wave equations for four functions $\psi^0,\ \psi^1,\ \psi^2,\ \psi^3$. That is why they can be used to describe particles with different spins $s = 0, 1/2, 1, 3/2, \ldots$. For system (1.3.1) to describe a field (particle) with the spin $s = 1/2$ it is necessary to impose an additional constraint (equation) on the function $\psi(x)$. Possible Poincaré-invariant additional conditions

$$\partial_\mu(\bar\psi\gamma_\mu\psi) = \lambda_1\bar\psi\psi + \lambda_2\bar\psi\gamma_4\psi + \lambda_3 \tag{1.3.2}$$

and

$$\bar\psi(i\gamma_\mu\partial_\mu - m)\psi = \lambda_1\bar\psi\psi + \lambda_2\bar\psi\gamma_4\psi + \lambda_3, \tag{1.3.3}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are constants, have been suggested in [91].

Nonlinear conditions (1.3.2), (1.3.3) select from the set of solutions of equation (1.3.1) the ones which correspond to a particle with the spin $s = 1/2$. On

the set of solutions of the system of PDEs (1.3.1), (1.3.2) the spinor representation of the Poincaré group having the generators (1.1.20) is realized.

It is interesting to note that the system of nonlinear equations (1.3.1), (1.3.2) with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ admits the group of nonlocal transformations

$$x'_\mu = x_\mu, \quad \psi'(x') = \psi(x) + \theta\gamma_4(i\gamma_\mu\partial_\mu - m)\psi(x),$$

where $\theta \in \mathbb{R}^1$ is a group parameter.

Another possibility of describing fields with spin $s = 1/2$ by the use of second-order equations is to consider a nonlinear equation of the form

$$(\partial_\mu\partial^\mu + m^2)\psi = R(\bar{\psi}, \psi, \underset{1}{\bar{\psi}}, \underset{1}{\psi}), \tag{1.3.4}$$

where $\underset{1}{\psi} = \Big\{\partial\psi^\alpha/\partial x_\mu, \ \alpha, \mu = 0, \ldots, 3\Big\}$, R is a four-component function.

The complete group-theoretical analysis of the above system can be carried out in the same way as it is done in Section 1.2. We will investigate symmetry properties of the important subclass of equations of the form (1.3.4)

$$\begin{aligned} \partial_\mu\partial^\mu + m^2\psi &= \Big\{F_1\Big(\partial_\mu(\bar{\psi}\psi), \partial_\mu(\bar{\psi}\gamma_4\psi), \bar{\psi}\psi, \bar{\psi}\gamma_4\psi\Big) \\ &\quad \times\gamma_\mu\partial_\mu + F_2(\bar{\psi}, \psi)\Big\}\psi. \end{aligned} \tag{1.3.5}$$

In (1.3.5) $F_1, \ F_2$ are variable (4×4)-matrices, $m = \text{const}$.

Theorem 1.3.1. *System of PDE (1.3.5) is invariant under the Poincaré group with the generators (1.1.20) iff*

$$\begin{aligned} F_1 &= g_1 + g_2\gamma_4 + (g_3 + g_4\gamma_4)\gamma \cdot v \\ &\quad +(g_5 + g_6\gamma_4)\gamma \cdot w + g_7\gamma \cdot v\gamma w, \end{aligned} \tag{1.3.6}$$

$$F_2 = f_1 + f_2\gamma_4, \tag{1.3.7}$$

where

$$\begin{aligned} &g_l = g_l(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi, v \cdot v, v \cdot w, w \cdot w), \quad l = 1, \ldots, 7, \\ &v_\mu = \partial_\mu(\bar{\psi}\psi), \quad w_\mu = \partial_\mu(\bar{\psi}\gamma_4\psi), \quad \mu = 0, \ldots, 3, \\ &f_i = f_i(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi), \ i = 1, 2 \end{aligned}$$

and $g_l, \ f_i$ are arbitrary smooth functions.

The proof is carried out with the help of the Lie method. First of all we note that system (1.3.5) admits the 4-parameter group of translations (1.1.24).

To obtain constraints on $F_1,\ F_2$ providing invariance of system (1.3.5) under the Lorentz group $O(1,3) \subset P(1,3)$ we act with the first prolongation of the operator J_{0a} given in (1.1.20) on the equation in question and pass to the set of its solutions. This procedure yields a system of determining equations for the matrix functions $F_1,\ F_2$. The system of PDEs for F_2 coincides with system (1.2.4) whose general solution is represented in the form (1.3.7).

On introducing the notations

$$Q_{0a} = v_0\partial_{v_a} + v_a\partial_{v_0} + w_0\partial_{w_a} + w_a\partial_{w_0}, \quad v_\mu = \partial_\mu(\bar\psi\psi), \quad w_\mu = \partial_\mu(\bar\psi\gamma_4\psi)$$

we rewrite the system of determining equations for F_1 in the form (1.2.4).

Expanding the (4×4)-matrix F_1 in the complete system of the Dirac matrices

$$F_1 = A + B_\mu\gamma^\mu + C_{\mu\nu}S^{\mu\nu} + D_\mu\gamma_4\gamma^\mu + E\gamma_4 \tag{1.3.8}$$

and substituting the expression obtained into (1.2.4) we arrive at the system of PDEs for the functions $A,\ B_\mu,\ \ldots, E$ of the type (1.2.6)–(1.2.9). Its general solution is given by the following formulae:

$$\begin{aligned} &A = g_1, \quad E = g_2, \quad B_\mu = g_3 v_\mu + g_5 w_\mu, \\ &D_\mu = g_4 v_\mu + g_6 w_\mu, \quad C_{\mu\nu} = g_7(v_\mu w_\nu - v_\nu w_\mu), \end{aligned} \tag{1.3.9}$$

where $g_1,\ g_2, \ldots, g_7$ are arbitrary smooth functions of the invariants of the group $O(1,3)$ $\ \bar\psi\psi,\ \bar\psi\gamma_4\psi,\ v\cdot v,\ v\cdot w,\ w\cdot w$.

Substitution of (1.3.9) into (1.3.8) gives rise to formula (1.3.6). The theorem is proved. $\triangleright$

Theorem 1.3.2. *System of PDEs (1.3.5) is invariant under the conformal group $C(1,3)$ with generators (1.1.22) iff*

$$\begin{aligned} F_1 &= (1/3)\gamma\cdot v(\bar\psi\psi)^{-1} + (h_1 + h_2\gamma_4)\Big\{\gamma\cdot v(\bar\psi\psi)^{-1} \\ &\quad -\gamma\cdot w(\bar\psi\gamma_4\psi)^{-1}\Big\} + \gamma_4(\bar\psi\psi)^{1/3}(h_3 + h_4\gamma_4), \\ F_2 &= (\bar\psi\psi)^{2/3}(\tilde f_1 + \tilde f_2\gamma_4), \quad m = 0. \end{aligned} \tag{1.3.10}$$

In (1.3.10) $h_1, \ldots, h_4$ are arbitrary smooth complex-valued functions of the invariants of the group $C(1,3)$ $(\bar\psi\psi)(\bar\psi\gamma_4\psi)^{-1},\ \{(\bar\psi\psi)^2 w\cdot w - 2v\cdot w(\bar\psi\psi)(\bar\psi\gamma_4\psi) + (\bar\psi\gamma_4\psi)^2 v\cdot v\}(\bar\psi\psi)^{-14/3}$ and $\tilde f_1,\ \tilde f_2$ are arbitrary smooth functions of $(\bar\psi\psi) \times(\bar\psi\gamma_4\psi)^{-1}$.

Proof. According to Theorem 1.3.1, the necessary and sufficient conditions for equation (1.3.5) to be invariant under the group $P(1,3) \subset C(1,3)$ are given

by equalities (1.3.6), (1.3.7). Acting by the first prolongation of the generator of the group of special conformal transformations $\theta_\mu K^\mu$, $\theta_\mu = \mathrm{const}$ on system of PDEs (1.3.5) with $F_1,\ F_2$ of the form (1.3.6), (1.3.7) and passing to the set of its solutions we obtain the system of PDEs for $A,\ B_1, \ldots, E,\ f_1,\ f_2$

$$\begin{aligned}
&L_1 g_1 = 2g_1, \quad L_2 g_1 = L_3 g_1 = 0,\\
&L_1 g_2 = 2g_2, \quad L_2 g_2 = L_3 g_2 = 0,\\
&L_1 g_j = -6g_j, \quad L_2 g_j = L_3 g_j = 0,\ j = 3, \ldots, 6,\\
&z_1 g_3 + z_2 g_5 = 1/3, \quad z_1 g_4 + z_2 g_6 = 0, \quad g_7 = 0,\\
&(z_1 \partial_{z_1} + z_2 \partial_{z_2} - 2/3) f_i = 0,\ i = 1, 2.
\end{aligned} \tag{1.3.11}$$

Here

$$\begin{aligned}
&L_1 = 6(z_1 \partial_{z_1} + z_2 \partial_{z_2}) + 16(z_3 \partial_{z_3} + z_4 \partial_{z_4} + z_5 \partial_{z_5}),\\
&L_2 = z_1 \partial_{z_5} + 2z_2 \partial_{z_4}, \quad L_3 = z_2 \partial_{z_5} + 2z_1 \partial_{z_3},\\
&z_1 = \bar\psi\psi, \quad z_2 = \bar\psi\gamma_4\psi, \quad z_3 = v\cdot v, \quad z_4 = v\cdot w, \quad z_5 = w\cdot w.
\end{aligned}$$

System of the first-order PDEs (1.3.11) is integrated in a standard way, its general solution having the form

$$\begin{aligned}
&g_1 = z_1^{1/3} h_3, \quad g_2 = z_1^{1/3} h_4, \quad g_3 = (1/3) z_1^{-1} + h_1 z_1^{-1},\\
&g_5 = -z_2^{-1} h_1, \quad g_4 = z_1^{-1} h_2, \quad g_6 = -z_2^{-1} h_2,\\
&f_1 = z_1^{2/3} \tilde f_1(z_1/z_2), \quad f_2 = z_1^{2/3} \tilde f_2(z_1/z_2),
\end{aligned}$$

where $h_1,\ h_2$ are arbitrary smooth complex-valued functions of $z_1 z_2^{-1},\ (z_1^2 z_5 + z_2^2 z_3 - 2z_1 z_2 z_4) z_1^{-14/3};\ \tilde f_i \in C^1(\mathbb{R}^1,\ \mathbb{C}^1)$.

Substitution of the above results into (1.3.6), (1.3.7) yields (1.3.10). The theorem is proved. $\rhd$

Consequence 1.3.1. *System of PDEs*

$$\{\partial_\mu \partial^\mu - F(\bar\psi, \psi)\}\psi = 0, \tag{1.3.12}$$

where F is a variable (4×4)-matrix, is not invariant with respect to the group $C(1,3)$.

The proof follows from the fact that the class of conformally-invariant equations (1.3.5), (1.3.10) does not contain equations of the form (1.3.12).

If we put in (1.3.10) $h_1 = h_2 = h_3 = h_4 = 0,\ \tilde f_1 = -\lambda^2 = \mathrm{const},\ \tilde f_2 = 0$, then the conformally-invariant second-order PDE

$$\Big\{\partial_\mu \partial^\mu - (1/3)(\bar\psi\psi)^{-1}\Big(\gamma_\mu \partial_\mu(\bar\psi\psi)\Big)\gamma_\nu \partial_\nu + \lambda^2 (\bar\psi\psi)^{2/3}\Big\}\psi = 0 \tag{1.3.13}$$

suggested in [155] is obtained. The direct verification shows that any solution of the Dirac-Gürsey equation satisfies PDE (1.3.13). That is why equation (1.3.13) as well as the Dirac-Gürsey equation can be used in conformally-invariant quantum field theories to describe a massless particle with the spin $s = 1/2$.

1.4. Symmetry of systems of nonlinear equations for spinor, vector and scalar fields

It is well-known (see, for example, [142]) that the classical electrodynamics equations

$$\begin{aligned} &(i\gamma_\mu\partial_\mu - e\gamma_\mu A^\mu)\psi = 0, \\ &\partial_\mu\partial^\mu A_\nu - \partial^\nu\partial_\mu A_\mu = -e\bar{\psi}\gamma_\nu\psi, \end{aligned} \tag{1.4.1}$$

where $A_\mu(x)$ is the vector-potential of electro-magnetic field, $e = \text{const}$, $\mu, \nu = 0, \ldots, 3$, are invariant under the conformal group $C(1,3)$ having the following generators:

$$\begin{aligned} P_\mu = \partial^\mu, \quad J_{\mu\nu} &= x_\mu P_\nu - x_\nu P_\mu + A_\mu\partial_{A^\nu} - A_\nu\partial_{A^\mu} \\ &- (1/2)\{\gamma_\mu\gamma_\nu\psi\}^\alpha\partial_{\psi^\alpha} + (1/2)\{\bar{\psi}\gamma_\mu\gamma_\nu\}^\alpha\partial_{\bar{\psi}^\alpha}, \ \mu \neq \nu, \end{aligned} \tag{1.4.2}$$

$$D = x_\mu\partial_\mu - (3/2)(\psi^\alpha\partial_{\psi^\alpha} + \bar{\psi}^\alpha\partial_{\bar{\psi}^\alpha}) - A_\mu\partial_{A_\mu}, \tag{1.4.3}$$

$$\begin{aligned} K_\mu = 2x_\mu D &- (x\cdot x)\partial^\mu - x_\mu(A_\nu\partial_{A_\nu} - \psi^\alpha\partial_{\psi^\alpha} - \bar{\psi}^\alpha\partial_{\bar{\psi}^\alpha}) \\ &- \{\gamma_\mu\gamma\cdot x\psi\}^\alpha\partial_{\psi^\alpha} - \{\bar{\psi}\gamma\cdot x\gamma_\mu\}^\alpha\partial_{\bar{\psi}^\alpha} + 2A_\mu x_\nu\partial_{A_\nu} \\ &- 2A\cdot x\partial_{A^\mu}. \end{aligned} \tag{1.4.4}$$

In formulae (1.4.2)–(1.4.4) $\partial_{A_\mu} = \partial/\partial A_\mu$, $\partial_{\psi^\alpha} = \partial/\partial\psi^\alpha$, $\partial_{\bar{\psi}^\alpha} = \partial/\partial\bar{\psi}^\alpha$; $\{\Psi\}^\alpha$ means the α-th component of the spinor Ψ; $\mu, \nu, \alpha = 0, \ldots, 3$.

Let us note that the operators K_μ (1.4.4) generate a 4-parameter group of special conformal transformations

$$\begin{aligned} &x'_\mu = (x_\mu - \theta_\mu x\cdot x)\sigma^{-1}(x), \\ &\psi'(x') = \sigma(x)(1 - \gamma\cdot\theta\,\gamma\cdot x)\psi(x), \\ &A'_\mu(x') = \{\sigma(x)g_{\mu\nu} + 2(x_\mu\theta_\nu - x_\nu\theta_\mu + 2\theta\cdot x\theta_\mu x_\nu \\ &\qquad - x\cdot x\theta_\mu\theta_\nu - \theta\cdot\theta x_\mu x_\nu)\}A^\nu(x), \end{aligned} \tag{1.4.5}$$

where $\sigma(x) = 1 - 2\theta \cdot x + (\theta \cdot \theta)(x \cdot x)$.

In [133, 142] another conformally-invariant system of PDEs for spinor and vector fields

$$\begin{aligned} &(i\gamma_\mu \partial_\mu - e\gamma_\mu A^\mu)\psi(x) = 0, \\ &\partial_\mu \partial^\mu A_\nu - \partial^\nu \partial_\mu A_\mu = \lambda A_\nu (A \cdot A) \end{aligned} \tag{1.4.6}$$

was suggested. A conjecture arises that there exist more general systems of nonlinear equations

$$\begin{aligned} &i\gamma_\mu \partial_\mu \psi - F(\bar{\psi},\, \psi,\, A) = 0, \\ &\partial_\mu \partial^\mu A_\nu - \partial^\nu \partial_\mu A_\mu = R_\nu(\bar{\psi},\, \psi,\, A) \end{aligned} \tag{1.4.7}$$

invariant under the conformal group.

In the present section we solve the problem of group-theoretical classification of systems of PDEs (1.4.7). Namely, we describe all functions $F = (F^0,\, F^1,\, F^2,\, F^3)^T$, R_μ such that system (1.4.7) is invariant with respect to the groups $P(1,3)$, $\widetilde{P}(1,3)$, $C(1,3)$.

In addition, symmetry analysis of systems of nonlinear equations for spinor and scalar fields

$$\begin{aligned} &i\gamma_\mu \partial_\mu \psi - F(u^*,\, u,\, \bar{\psi},\, \psi) = 0, \\ &\partial_\mu \partial^\mu u - H(u^*,\, u,\, \bar{\psi},\, \psi) = 0; \end{aligned} \tag{1.4.8}$$

vector and scalar fields

$$\begin{aligned} &\partial_\mu \partial^\mu u - H(u^*,\, u,\, A) = 0, \\ &\partial_\mu \partial^\mu A_\nu - \partial^\nu \partial_\mu A_\mu = R_\nu(u^*,\, u,\, A) \end{aligned} \tag{1.4.9}$$

is carried out.

In (1.4.8), (1.4.9) $F = (F^0,\, F^1,\, F^2,\, F^3)^T$; F^μ, H, R_μ are some smooth functions; $u(x) \in C^2(\mathbb{R}^4, \mathbb{C}^1)$.

Theorem 1.4.1. *System (1.4.8) is invariant under*

1) the Poincaré group iff

$$F = (f_1 + f_2\gamma_4)\psi, \quad H = h(u^*,\, u,\, \bar{\psi}\psi,\, \bar{\psi}\gamma_4\psi), \tag{1.4.10}$$

where f_1, f_2 are arbitrary smooth complex-valued functions of u^*, u, $\bar{\psi}\psi$, $\bar{\psi}\gamma_4\psi$;

2) the extended Poincaré group $\widetilde{P}(1,3) = P(1,3) \otimes D(1)$, where $D(1)$ is the one-parameter group of scale transformations

$$x'_\mu = x_\mu e^\theta, \quad u' = ue^{-k_2\theta}, \quad \psi' = \psi e^{-k_1\theta}, \quad \theta,\ k_1,\ k_2 = \text{const}, \tag{1.4.11}$$

iff $F,\ H$ *are given by (1.4.10) with*

$$\begin{aligned}&f_i=(\bar\psi\psi)^{1/2k_1}\tilde f_i(w_1,\,w_2,\,w_3),\quad h=(u^*u)^{1/k_2}u\tilde h(w_1,\,w_2,\,w_3),\\&w_1=u/u^*,\quad w_2=u^{2k_1}(\bar\psi\psi)^{-k_2},\quad w_3=u^{2k_1}(\bar\psi\gamma_4\psi)^{-k_2},\\&\{\tilde f_i,\ \tilde h\}\subset C^1(\mathbb{C}^3,\mathbb{C}^1),\quad i=1,2;\end{aligned}\tag{1.4.12}$$

3) the conformal group $C(1,3)=P(1,3)\otimes D(1)\otimes K(1,3)$, *where* $D(1)$ *is given by (1.4.11) with* $k_1=3/2,\ k_2=1$ *and the 4-parameter group of special conformal transformations* $K(1,3)$ *has the form*

$$\begin{aligned}&x'_\mu=(x_\mu-\theta_\mu x\cdot x)\sigma^{-1}(x),\\&\psi'(x')=\sigma(x)(1-\gamma\cdot\theta\gamma\cdot x)\psi(x),\\&u'(x')=\sigma(x)u(x),\end{aligned}\tag{1.4.13}$$

iff $F,\ H$ *are given by formulae (1.4.10), (1.4.12) with* $k_1=3/2,\ k_2=1$*;*

4) the group $C(1,3)\otimes U(1)$, *where* $U(1)$ *is the one-parameter group of gauge transformations*

$$x'_\mu=x_\mu,\quad \psi'(x)=e^{i\theta}\psi(x),\quad u'(x)=e^{i\theta}u(x),\quad \theta\in\mathbb{R}^1,$$

iff

$$\begin{aligned}&F=(\bar\psi\psi)^{1/3}\{\tilde f_1(z_1,\,z_2)+\gamma_4\tilde f_2(z_1,\,z_2)\}\psi,\\&H=|u|^2u\tilde h(z_1,\,z_2),\quad \{\tilde f_1,\ \tilde f_2,\ \tilde h\}\subset C^1(\mathbb{R}^2,\mathbb{C}^1),\\&z_1=\bar\psi\psi|u|^{-3},\quad z_2=\bar\psi\gamma_4\psi|u|^{-3}.\end{aligned}\tag{1.4.14}$$

The proof is carried out with the help of the Lie method. Acting on system of PDEs (1.4.8) by the first prolongation of the operator J_{0a} (1.1.20) and passing to the set of its solutions we get necessary and sufficient conditions of Lorentz invariance of system (1.4.8) in the form

$$Q_{0a}\Phi-(1/2)[\Phi,\,\gamma_0\gamma_a]=0,\quad Q_{0a}H=0,\quad a=1,2,3,\tag{1.4.15}$$

where $Q_{0a}=-(1/2)\{\gamma_0\gamma_a\psi\}^\alpha\partial_{\psi^\alpha}+(1/2)\{\bar\psi\gamma_0\gamma_a\}^\alpha\partial_{\bar\psi^\alpha}$, $\Phi=\Phi(u^*,\,u,\,\bar\psi,\,\psi)$ is a (4×4)-matrix (we have represented the four-component function F in the form $\Phi\psi$).

Since the first equation of system (1.4.15) coincides with (1.2.4) and the second one with (1.2.6), we can write down their general solutions using the results obtained in Section 1.2. According to (1.2.2), (1.2.11) the general solution of system of PDEs (1.4.15) has the form (1.4.10). Taking into account

the fact that system (1.4.8) is invariant under the 4-parameter group of translations (1.1.24) we arrive at the assertion 1 of Theorem 1.4.1.

Acting on system of PDEs (1.4.8), (1.4.10) by the first prolongation of the generator of the group of scale transformations

$$D = x_\mu \partial_\mu - k_1 \psi^\alpha \partial_{\psi^\alpha} - k_1 \bar{\psi}^\alpha \partial_{\bar{\psi}^\alpha} - k_2 u \partial_u - k_2 u^* \partial_{u^*}$$

and passing to the set of its solutions we get the following system of PDEs for $f_1,\ f_2,\ h$:

$$\begin{aligned}
&k_2(\rho_1 f_{i\rho_1} + \rho_2 f_{i\rho_2}) + 2k_1(\rho_3 f_{i\rho_3} + \rho_4 f_{i\rho_4}) = 1, \quad i = 1, 2,\\
&k_2(\rho_1 h_{\rho_1} + \rho_2 h_{\rho_2}) + 2k_1(\rho_3 h_{\rho_3} + \rho_4 h_{\rho_4}) = 2,\\
&f_{i\rho_n} = \partial f_i / \partial \rho_n, \quad h_{\rho_n} = \partial h / \partial \rho_n, \quad n = 1, \ldots, 4,
\end{aligned}$$

where $\rho_1 = u^*,\ \rho_2 = u,\ \rho_3 = \bar{\psi}\psi,\ \rho_4 = \bar{\psi}\gamma_4\psi$ is a complete system of functionally-independent invariants of the group $P(1,3)$. General solution of the above system is given by the formulae (1.4.12), $w_1,\ w_2,\ w_3$ being a complete system of functionally-independent invariants of the extended Poincaré group. Since the conformal group contains the group $\widetilde{P}(1,3)$, the requirement of $C(1,3)$-invariance of system of PDEs (1.4.8) leads to formulae (1.4.10), (1.4.12) under $k_1 = 3/2,\ k_2 = 1$. The sufficiency of assertion 3 is established by direct verification.

To select from the class of conformally-invariant equations of the form (1.4.8) the equations which admit the group $U(1)$ we act with the first prolongation of the generator of this group on system (1.4.8) and pass to the set of its solutions. As a result, we have

$$\begin{aligned}
&2w_1 \tilde{f}_{iw_1} + 3w_2 \tilde{f}_{iw_2} + 3w_3 \tilde{f}_{iw_3} = 0, \quad i = 1, 2,\\
&2w_1 \tilde{h}_{w_1} + 3w_2 \tilde{h}_{w_2} + 3w_3 \tilde{h}_{w_3} = 0.
\end{aligned}$$

General solution of the above equations is represented in the form

$$\tilde{f}_i = \tilde{f}_i(w_1^{3/2} w_2^{-1},\, w_1^{3/2} w_3^{-1}), \quad \tilde{h} = \tilde{h}(w_1^{3/2} w_2^{-1},\, w_1^{3/2} w_3^{-1}).$$

Putting

$$w_1 = u(u^*)^{-1}, \quad w_2 = u^3(\bar{\psi}\psi)^{-1}, \quad w_3 = u^3(\bar{\psi}\gamma_4\psi)^{-1}$$

yields formulae (1.4.14). The theorem is proved. $\triangleright$

Note 1.4.1. In [90] a model for description of interaction of spinor and real-valued scalar fields based on the relativistic Hamilton equation

$$\begin{aligned} &i\gamma_\mu \partial_\mu \psi - F(u,\, \bar\psi,\, \psi) = 0, \\ &(\partial_\mu u)(\partial^\mu u) = H(u,\, \bar\psi,\, \psi) \end{aligned} \tag{1.4.16}$$

was suggested. Using the Lie method we can prove that system of PDEs (1.4.16) admits the Poincaré group iff

$$\begin{aligned} &F = \{f_1(u,\, \bar\psi\psi,\, \bar\psi\gamma_4\psi) + \gamma_4 f_2(u,\, \bar\psi\psi,\, \bar\psi\gamma_4\psi)\}\psi, \\ &H = h(u,\, \bar\psi\psi,\, \bar\psi\gamma_4\psi). \end{aligned} \tag{1.4.17}$$

Provided

$$f_i = (\bar\psi\psi)^{1/2k_1} \tilde f_i(w_1,\, w_2), \quad h = u^{2(k_2+1)/k_2} \tilde h(w_1,\, w_2), \quad i = 1, 2,$$

where $w_1 = u^{2k_1}(\bar\psi\psi)^{-k_2}$, $w_2 = u^{2k_1}(\bar\psi\gamma_4\psi)^{-k_2}$, system of PDEs (1.4.16) is invariant with respect to the extended Poincaré group.

The next two theorems are given without proof.

Theorem 1.4.2. *System of nonlinear equations (1.4.7) is invariant under*

1) the Poincaré group with generators (1.4.2) iff

$$\begin{aligned} &F(\bar\psi,\, \psi,\, A) = \{\gamma\cdot A f_1 + \gamma_4\gamma\cdot A f_2 + f_3 + \gamma_4 f_4\}\psi, \\ &R_\mu(\bar\psi,\, \psi,\, A) = A_\mu g_1 + \bar\psi\gamma_\mu\psi g_2 + \bar\psi\gamma_4\gamma_\mu\psi g_3 + \psi^T\gamma_0\gamma_2\gamma_\mu\psi g_4, \end{aligned} \tag{1.4.18}$$

where f_i are arbitrary complex-valued functions and h_i are arbitrary real-valued functions of

$$\bar\psi\psi, \quad \bar\psi\gamma_4\psi, \quad \bar\psi\gamma\cdot A\psi, \quad \bar\psi\gamma_4\gamma\cdot A\psi, \quad \psi^T\gamma_0\gamma_2\gamma\cdot A\psi, \quad A\cdot A;$$

2) the extended Poincaré group $\widetilde P(1,3)$ with generators (1.4.2) and

$$D = x_\mu\partial_\mu - k_1\psi_\alpha\partial_{\psi_\alpha} - k_1\psi_\alpha\partial_{\psi_\alpha} - k_2 A_\mu\partial_{a_\mu}, \quad \{k_1, k_2\} \subset \mathbb{R}^1$$

iff functions F, R_μ are given by formulae (1.4.18), where

$$\begin{aligned} &f_i = (A\cdot A)^{(1-k_2)/2k_2} \tilde f_i, \quad i = 1, 2, \\ &f_j = (\bar\psi\psi)^{1/2k_1} \tilde f_j, \quad j = 3, 4, \quad g_1 = (A\cdot A)^{1/k_2} \tilde g_1, \\ &g_i = (\bar\psi\psi)^{(k_2-2k_1+2)/2k_1} \tilde g_i, \quad i = 2, 3, 4, \end{aligned} \tag{1.4.19}$$

$\tilde{f}_1, \ldots, \tilde{f}_4,\ \tilde{g}_1, \ldots, \tilde{g}_4$ *being arbitrary smooth functions of*

$$(\bar{\psi}\psi)(\bar{\psi}\gamma_4\psi)^{-1}, \quad (\bar{\psi}\psi)^{k_2}(A\cdot A)^{-k_1}, \quad (\bar{\psi}\psi)^{2k_1+k_2}(\bar{\psi}\gamma\cdot A\psi)^{-2k_1},$$
$$(\bar{\psi}\psi)^{2k_1+k_2}(\bar{\psi}\gamma_4\gamma\cdot A\psi)^{-2k_1}, \quad (\bar{\psi}\psi)^{2k_1+k_2}(\psi^T\gamma_0\gamma_2\gamma\cdot A\psi)^{-2k_1};$$

3) the group $C(1,3)$ with generators (1.4.2)–(1.4.4) iff the functions F, R_μ are given by (1.4.18), (1.4.19) under $k_1 = 3/2$, $k_2 = 1$;

4) the group $C(1,3)\otimes U(1)$, where $U(1)$ is the group of gauge transformations

$$\begin{aligned} &x'_\mu = x_\mu, \quad \psi' = \psi e^{i\lambda\theta(x)},\\ &A'_\mu = A_\mu + \partial^\mu\theta(x), \quad \theta(x)\in C^3(\mathbb{R}^4,\mathbb{R}^1), \end{aligned} \tag{1.4.20}$$

iff

$$\begin{aligned} &F(\bar{\psi},\,\psi,\,A) = \{\lambda\gamma\cdot A + f_1(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}) + \gamma_4 f_2(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1})\}\psi,\\ &R_\mu(\bar{\psi},\,\psi,\,A) = \bar{\psi}\gamma_\mu\psi g_1(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}) + \bar{\psi}\gamma_4\gamma_\mu\psi g_2(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}), \end{aligned} \tag{1.4.21}$$

where $f_i\in C^1(\mathbb{R}^1,\mathbb{C}^1)$, $g_i\in C^1(\mathbb{R}^1,\mathbb{R}^1)$, $i=1,2$, $\lambda = \mathrm{const}$.

Consequence 1.4.1. *On the set of solutions of system of PDEs (1.4.7), (1.4.21) an infinite-dimensional representation of the Lie algebra $AC(1,3)$ is realized, basis elements of the algebra having the form*

$$\begin{aligned} &\widetilde{P}_\mu = P_\mu, \quad \widetilde{J}_{\mu\nu} = J_{\mu\nu}, \quad \widetilde{D} = D + i\lambda(\psi^\alpha\partial_{\psi^\alpha} - \bar{\psi}^\alpha\partial_{\bar{\psi}^\alpha}),\\ &\widetilde{K}_\mu = K_\mu + \partial_{A_\mu} + i\lambda x_\mu(\psi^\alpha\partial_{\psi^\alpha} - \bar{\psi}_\alpha\partial_{\bar{\psi}^\alpha}), \end{aligned} \tag{1.4.22}$$

where the operators P_μ, $J_{\mu\nu}$, D, K_μ are given by (1.4.4).

The proof is reduced to verification of the commutation relations of the algebra $AC(1,3)$ if we note that the operators

$$\begin{aligned} &Q_1 = i\lambda(\psi^\alpha\partial_{\psi^\alpha} - \bar{\psi}^\alpha\partial_{\bar{\psi}^\alpha}),\\ &Q_{2\mu} = i\lambda x_\mu(\psi^\alpha\partial_{\psi^\alpha} - \bar{\psi}^\alpha\partial_{\bar{\psi}^\alpha}) + \partial_{A_\mu}, \quad \mu = 0,\ldots,3 \end{aligned} \tag{1.4.23}$$

generate transformation groups of the form (1.4.20). ▷

Thus, system (1.4.7), (1.4.21) possesses a dual conformal symmetry. To fix a definite representation of $AC(1,3)$ it is necessary to impose an additional constraint on the vector field $A_\mu(x)$. In [82] the nonlinear equation

$$\partial_\mu(A_\mu A\cdot A) = 0 \tag{1.4.24}$$

invariant under the algebra (1.4.4) was suggested. Since PDE (1.4.24) is not invariant under transformation groups generated by the operators $Q_{2\mu}$ from (1.4.23), it does not admit the 4-parameter group with generators $\widetilde{K}_\mu$ from (1.4.22). Consequently, system of PDEs (1.4.7), (1.4.21), (1.4.24) is invariant under the conformal algebra (1.4.22).

Analogously, using results obtained in the paper [142] we conclude that system (1.4.7), (1.4.21) supplemented by the additional condition

$$\partial_\mu A_\mu - 2A \cdot A = 0 \tag{1.4.25}$$

is invariant under the conformal algebra (1.4.22) and is not invariant under the algebra (1.4.4).

Theorem 1.4.3. *System of nonlinear PDEs (1.4.9) is invariant under*

1) the Poincaré group iff

$$H = h(u^*,\, u,\, A \cdot A), \quad R_\mu = A_\mu g(u^*,\, u,\, A \cdot A),$$

where $h \in C^1(\mathbb{C}^2 \times \mathbb{R}^1, \mathbb{C}^1)$, $g \in C^1(\mathbb{C}^2 \times \mathbb{R}^1, \mathbb{R}^1)$;

2) the extended Poincaré group $\widetilde{P}(1,3) = P(1,3) \otimes D(1)$, *where* $D(1)$ *is a one-parameter group of scale transformations*

$$\begin{aligned} &x'_\mu = x_\mu e^\theta, \quad A'_\mu = A_\mu e^{-k_1\theta}, \\ &u' = ue^{-k_2\theta}, \quad u^{*\prime} = u^* e^{-k_2\theta}, \quad \theta \in \mathbb{R}^1, \end{aligned} \tag{1.4.26}$$

iff

$$\begin{aligned} &H = |u|^{2/k_2} u h\Big(u^* u^{-1},\, |u|^{-2k_1}(A \cdot A)^{k_2}\Big), \\ &R_\mu = (A \cdot A)^{1/k_1} g\Big(u^* u^{-1},\, |u|^{-2k_1}(A \cdot A)^{k_2}\Big) A_\mu; \end{aligned} \tag{1.4.27}$$

3) the conformal group $C(1,3) = P(1,3) \otimes D(1) \otimes K(1,3)$, *where* $D(1)$ *is the group (1.4.26) with* $k_1 = 1$, $k_2 = 1$ *and* $K(1,3)$ *is the 4-parameter group of special conformal transformations*

$$\begin{aligned} &x'_\mu = (x_\mu - \theta_\mu x \cdot x)\sigma^{-1}(x), \quad u' = \sigma(x) u, \\ &A'_\mu = \{\sigma(x) g_{\mu\nu} + 2(x_\mu \theta_\nu - x_\nu \theta_\mu + 2\theta \cdot x \theta_\mu x_\nu \\ &\quad - x \cdot x \theta_\mu \theta_\nu - \theta \cdot \theta x_\mu x_\nu)\} A^\nu, \end{aligned} \tag{1.4.28}$$

where $s(x) = 1 - 2\theta \cdot x + \theta \cdot \theta x \cdot x$, $\theta_\mu = \mathrm{const}$, $\mu = 0, \ldots, 3$, *iff* H, R_μ *are of the form (1.4.27) under* $k_1 = k_2 = 1$;

4) the group $C(1,3) \otimes U(1)$, where $U(1)$ is the group of gauge transformations

$$x'_\mu = x_\mu, \quad A'_\mu = A_\mu, \quad u' = ue^{i\theta}, \quad \theta \in \mathbb{R}^1,$$

iff

$$\begin{aligned}&H = |u|^2 uh(|u|^{-2} A \cdot A), \quad R_\mu = A_\mu A \cdot Ag(|u|^{-2} A \cdot A),\\ &h \in C^1(\mathbb{R}^1, \mathbb{C}^1), \quad g \in C^1(\mathbb{R}^1, \mathbb{R}^1).\end{aligned}$$

Thus, using the symmetry selection principle we narrow substantially classes of physically admissible nonlinear generalizations of the Maxwell-Dirac, Dirac-d'Alembert and Maxwell-d'Alembert equations.

1.5. Conditional symmetry and reduction of partial differential equations

Analyzing already known methods of construction of exact solutions of nonlinear partial differential equations we come to conclusion that a majority of them is based on the idea of narrowing the set of solutions, i.e., selecting from the whole set of solutions specific subsets which admit analytic description. To implement this idea we have to impose some additional constraints (equations) on the set of solutions of the equation under consideration selecting such subsets. Clearly, additional equations are supposed to be simpler than the initial one. Supplementing the initial equation with additional conditions we come, as a rule, to an over-determined system of PDEs. So there arises a problem of investigating the matter of its compatibility.

To clarify the above points we will consider an instructive example. Let

$$U(x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) = 0 \tag{1.5.1}$$

be a second-order PDE with two independent variables $x_0,\ x_1$ which does not depend explicitly on x_0.

Since coefficients of PDE (1.5.1) do not contain the variable x_0, substitution of the expression

$$u = \varphi(x_1) \tag{1.5.2}$$

into (1.5.1) results in a differential equation containing $x_1,\ \varphi,\ \dot{\varphi},\ \ddot{\varphi}$ only, i.e.,

$$\widetilde{U}(x_1,\, \varphi,\, \dot{\varphi},\, \ddot{\varphi}) = 0. \tag{1.5.3}$$

Consequently, using the fact that PDE (1.5.1) does not contain the variable x_0 we *reduce* it to an ODE assuming that a particular solution also does not depend on x_0.

But from the group-theoretical point of view the independence of PDE (1.5.1) of x_0 means that it is invariant under the one-parameter translation group with respect to the variable x_0

$$x_0' = x_0 + \theta, \quad x_1' = x_1, \quad u' = u, \quad \theta \in \mathbb{R}^1 \tag{1.5.4}$$

having the generator $X = \partial_{x_0}$. And what is more, formula (1.5.2) defines the most general manifold in the three-dimensional space of variables $x_0,\ x_1,\ u$ which is invariant with respect to the above group. Expression (1.5.2) is called a solution (an Ansatz) invariant under the one-parameter group (1.5.4).

The above said can be summarized in the form of the following statement: a solution invariant under the group of translations (1.5.4) reduces PDE admitting the same group to ODE. When generalized to the case of an arbitrary admissible one-parameter group, this statement plays a key role in applications of Lie transformation groups to construction of exact solutions of mathematical physics equations.

The way for obtaining an invariant solution is entirely algorithmic. Since we are looking for a manifold $u = f(x_0, x_1)$ which does not contain explicitly the variable x_0 (is invariant with respect to the group (1.5.4)) we should require that $\partial f/\partial x_0 = 0$. Consequently, to find a solution of PDE (1.5.1) invariant under the group (1.5.4) it is necessary to solve an over-determined system of PDEs

$$U(x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) = 0, \quad u_{x_0} = 0.$$

We have paid so much attention to a very simple example, since it gives an adequate illustration to ideas of the symmetry reduction method pioneered by Sophus Lie. Moreover, a general case of PDE

$$U(x_0,\, x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) = 0 \tag{1.5.5}$$

invariant under a one-parameter transformation group having a generator $X = \xi_0(x,u)\partial_{x_0} + \xi_1(x,u)\partial_{x_1} + \eta(x,u)\partial_u$ is reduced to the particular case considered above. Indeed, it is known from the general theory of PDEs that there is a change of variables

$$\tilde{x}_0 = F_0(x,u), \quad \tilde{x}_1 = F_1(x,u), \quad \tilde{u} = G(x,u)$$

transforming the operator X to the form $X' = \partial_{\tilde{x}_0}$. Consequently, PDE (1.5.5) after being rewritten in the variables $\tilde{x}$, $\tilde{u}$ is invariant under the one-parameter transformation group with the generator $X' = \partial_{\tilde{x}_0}$, i.e., under the group (1.5.4). According to the above proved a substitution $\tilde{u} = \varphi(\tilde{x}_1)$ reduces the equation transformed to ODE for a function φ. Hence, we conclude that the substitution $F_0(x,u) = \varphi\Big(F_1(x,u)\Big)$ reduces the initial equation to ODE.

Thus, given a one-parameter transformation group admitted by partial differential equation (1.5.5), we can reduce it to an ODE by means of a substitution of a special form (invariant solution or Ansatz)

$$u = f\Big(x,\, \varphi(\omega(x,u))\Big), \tag{1.5.6}$$

where f, ω are some functions determined by the form of the generator of the group. A natural question arises: do invariant solutions exhaust the set of substitutions (1.5.6) reducing given PDE to an ODE? A negative answer to this question has led us to the notion of *conditional symmetry* of partial differential equations.

The notion and terminology of conditional symmetry of PDEs was introduced for the first time in [91, 92, 116] and developed in a series of papers and monographs [96, 97], [105]–[107], [120, 124, 108, 126, 127, 128, 137, 143], [154]–[160], [246, 303, 308] (see also [32, 52, 211, 234]). The principal idea of conditional symmetry of PDE is illustrated by the following example. The equation

$$U(x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) + V(x_0,\, x_1,\, u,\, \underset{1}{u},\, \underset{2}{u})u_{x_0} = 0, \quad \partial V/\partial_{x_0} \neq 0$$

is not invariant under the translations with respect to x_0. Nevertheless, Ansatz (1.5.2) invariant under the translation group (1.5.4) reduces it to an ODE. An explanation for this phenomenon is quite simple. The matter is that the second "non-invariant" term of the equation in question vanishes on the manifold (1.5.2). Saying it in another way, the system of two PDEs

$$U(x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) + V(x_0,\, x_1,\, u,\, \underset{1}{u},\, \underset{2}{u})u_{x_0} = 0, \quad u_{x_0} = 0 \tag{1.5.7}$$

is invariant under the group (1.5.4).

Consequently, from the point of view of reducibility of PDE (1.5.5) by means of the Ansatz invariant under the one-parameter transformation group with the generator $X = \xi_0(x,u)\partial_{x_0} + \xi_1(x,u)\partial_{x_1} + \eta(x,u)\partial_u$ it is enough to require the invariance of a constrained system of PDEs

$$U(x_0,\, x_1,\, u,\, \underset{1}{u},\, \underset{2}{u}) = 0, \quad \xi_0(x,u)u_{x_0} + \xi_1(x,u)u_{x_1} - \eta(x,u) = 0.$$

This is a source of the term conditional symmetry. Equation (1.5.5) is non-invariant with respect to the group having the generator X but being taken together with a condition $Xu = 0$ it admits the mentioned group. Consequently, it is *conditionally-invariant* under the Lie group with the generator X.

1. Reduction of PDEs. Consider an over-determined system of PDEs of the form

$$U_A(x, u, \underset{1}{u}, \dots, \underset{r}{u}) = 0, \quad A = 1, \dots, M, \tag{1.5.8}$$

$$\xi_{a\mu}(x, u) u^\alpha_{x_\mu} - \eta^\alpha_a(x, u) = 0, \quad a = 1, \dots, N, \tag{1.5.9}$$

where $x = (x_0, x_1, \dots, x_{n-1}),\ u = (u^0, u^1, \dots, u^{m-1}),$

$$\underset{s}{u} = \{\partial^s u^\alpha / \partial x_{\mu_1} \dots \partial x_{\mu_s},\ 0 \le \alpha \le m-1,\ 0 \le \mu_i \le n-1\},$$

$U_A,\ \xi_{a\mu},\ \eta^\alpha$ are smooth enough functions, $N \le n-1$. In the following, we suppose that the condition

$$\operatorname{rank} \|\xi_{a\mu}(x, u)\|^{N\ \ n-1}_{a=1\mu=0} = N \tag{1.5.10}$$

holds.

Definition 1.5.1. Set of the first-order differential operators

$$Q_a = \xi_{a\mu}(x, u)\partial_{x_\mu} + \eta^\alpha_a(x, u)\partial_{u^\alpha}, \tag{1.5.11}$$

where $\xi_{a\mu},\ \eta^\alpha_a$ are smooth functions, is called involutive if there exist such smooth functions $f^c_{ab}(x, u)$ that

$$[Q_a,\, Q_b] = f^c_{ab} Q_c, \quad a, b, c = 1, \dots, N. \tag{1.5.12}$$

The simplest example of an involutive set of operators is given by first-order differential operators forming a Lie algebra. In such a case $f^c_{ab} = \text{const}$, $a, b, c = 1, \dots, N$ are called structure constants of the Lie algebra.

It is common knowledge that conditions (1.5.12) are necessary and sufficient for the system of PDEs (1.5.9) to be compatible (the Frobenius theorem [250]). Its general solution can be represented in the form

$$F^\alpha(\omega_1, \omega_2, \dots, \omega_{n+m-N}) = 0, \quad \alpha = 0, \dots, m-1, \tag{1.5.13}$$

where $F^\alpha \in C^1(\mathbb{C}^{n+m-N}, \mathbb{C}^1)$ are arbitrary functions, $\omega_i = \omega_i(x, u)$ are functionally-independent first integrals of system of PDEs (1.5.9).

Due to condition (1.5.10) we can choose m first integrals $\omega_{j_1},\ldots,\omega_{j_m}$ satisfying the condition $\det\|\partial\omega_{j_i}/\partial u^\alpha\|^{m}_{i=1}{}^{m-1}_{\beta=0}\neq 0$, since otherwise integrals ω_1, ω_2, …, ω_{n+m-N} would be functionally-dependent.

Changing, if necessary, numeration we can put $j_i=i$ and thus get m first integrals $\omega_1,\ldots,\omega_m$ of the system of PDEs (1.5.9) satisfying the following condition:

$$\det\|\partial\omega_i/\partial u^\alpha\|^{m}_{i=1}{}^{m-1}_{\beta=0}\neq 0. \tag{1.5.14}$$

Resolving relations (1.5.13) with respect to $\omega_1,\ldots,\omega_m$ we have

$$\omega_i=\varphi_i(\omega_{m+1},\ldots,\omega_{n+m-N}), \tag{1.5.15}$$

where $\varphi_i\in C^1(\mathbb{C}^{n-N},\mathbb{C}^1)$, $i=1,\ldots,m$ are arbitrary functions.

Definition 1.5.2. Expression (1.5.15) is called an Ansatz for the field $u^\alpha=u^\alpha(x)$ invariant under the set of operators (1.5.11) provided (1.5.14) holds.

Formulae (1.5.15) take an especially simple and clear form provided

$$\xi_{a\mu}=\xi_{a\mu}(x),\quad \eta^\alpha_a=A^{\alpha\beta}_a(x)u^\beta,\quad a=1,\ldots,N,\quad \alpha=0,\ldots,m-1. \tag{1.5.16}$$

Given the condition (1.5.16) operators (1.5.11) are rewritten in a non-Lie form

$$Q_a=\xi_{a\mu}(x)\partial_{x_\mu}+\eta_a(x),\quad a=1,\ldots,N, \tag{1.5.17}$$

where $\eta_a=\|-A^{\alpha\beta}_a(x)\|^{m-1}_{\alpha,\beta=0}$ are $(m\times m)$-matrices and system (1.5.9) is rewritten as a system of linear PDEs

$$\xi_{a\mu}(x)u_{x_\mu}+\eta_a(x)u=0,\quad a=1,\ldots,N. \tag{1.5.18}$$

Here $u=(u^0,\ldots,u^{m-1})^T$.

Lemma 1.5.1. *Assume that conditions (1.5.10), (1.5.16) hold. Then, a set of functionally-independent first integrals of system of PDEs (1.5.9) can be chosen as follows*

$$\omega_i=b^\alpha_i(x)u^\alpha,\quad i=1,\ldots,m,$$
$$\omega_{m+j}=\omega_{m+j}(x),\quad j=1,\ldots,n-N$$

and besides $\det\|b^\alpha_i(x)\|^{m}_{i=1}{}^{m-1}_{\alpha=0}\neq 0$.

Proof. Consider the following system of matrix PDEs:

$$\xi_{a\mu}(x)F_{x_\mu}=F\eta_a(x), \tag{1.5.19}$$

where $F = \|f^{\alpha\beta}(x)\|_{\alpha,\beta=0}^{m-1}$ is an $(m \times m)$-matrix and $\xi_a,\ \eta_a = \| - A_a^{\alpha\beta}(x)\|_{\alpha,\beta=0}^{m-1}$ are coefficients of the operators Q_a. Since the operators Q_a form an involutive set, the above system is compatible and its general solution has the form

$$F(x) = \Theta B(x),$$

where Θ is an $(m \times m)$-matrix whose elements are arbitrary functions of a complete set of functionally-independent first integrals of the system

$$\xi_{a\mu}\partial_\mu \omega = 0, \quad a = 1, \ldots, N \tag{1.5.20}$$

and $B(x) = \|b_i^\alpha(x)\|_{i,\alpha=0}^{m-1}$ is a particular solution of (1.5.19) with $\det B(x) \neq 0$.

It is straightforward to check that from the involutivity of the set of operators Q_a it follows that the operators $Q'_a = \xi_{a\mu}\partial_\mu$ form an involutive set. Consequently, system (1.5.20) is compatible and what is more due to the condition (1.5.10) the number of its functionally-independent first integrals is equal to $n - N$. We denote these as: $\omega_{m+1}(x)$, $\omega_{m+2}(x)$,$\ldots$,$\omega_{n+m-N}(x)$.

As $\det \|b_i^\alpha(x)\|_{i,\alpha=0}^{m-1} \neq 0$, the expressions $b_1^\alpha(x)u^\alpha$, $\ldots$, $b_m^\alpha(x)u^\alpha$, $\omega_{m+1}(x)$, $\ldots$, $\omega_{m+n-N}(x)$ are functionally-independent. If we prove that the functions $b_i^\alpha(x)u^\alpha$, $i = 1, \ldots, m$ are first integrals, the proof of the lemma will be completed.

Acting by the operators Q_a on the functions $b_i^\alpha(x)u^\alpha$ one has

$$\Big(\xi_{a\mu}\partial_\mu + A_a^{\gamma\beta}(x)u^\beta\partial_{u^\gamma}\Big)\Big(b_i^\alpha(x)u^\alpha\Big) = \Big(\xi_{a\mu}\partial_\mu b_i^\beta(x) + b_i^\alpha(x)A_a^{\alpha\beta}(x)\Big)u^\beta = 0$$

(we have taken into account that the matrix $B(x) = \|b_i^\alpha(x)\|_{i,\alpha=0}^{m-1}$ satisfies (1.5.19)) the same which is required. The lemma is proved. $\triangleright$

Due to Lemma 1.5.1 we can resolve formulae (1.5.15) with respect to u^α and thus transform an Ansatz invariant under operators (1.5.17) to the form

$$u^\alpha = a^{\alpha\beta}(x)\varphi^\beta(\omega_{m+1}, \ldots, \omega_{m+n-N})$$

or (in the matrix notation)

$$u = A(x)\varphi(\omega_{m+1}, \ldots, \omega_{m+n-N}), \tag{1.5.21}$$

where $A(x) = \|a_{\alpha\beta}(x)\|_{\alpha,\beta=0}^{m-1}$ is the inverse of the matrix B.

Since the matrix function $B(x)$ satisfies the system of PDEs (1.5.19), the following equalities hold

$$\begin{aligned}\xi_{a\mu}\partial_\mu A(x) &= \xi_{a\mu}\partial_\mu B^{-1}(x) = -B^{-1}(x)\Big(\xi_{a\mu}\partial_\mu B(x)\Big)B^{-1}(x)\\ &= -B^{-1}(x)B(x)\eta_a B^{-1}(x) = -\eta_a A(x).\end{aligned}$$

Consequently, we have established that the Ansatz invariant under the involutive set of operators (1.5.17) satisfying condition (1.5.10) is represented in the form (1.5.21), where $A(x)$ is a nonsingular $(m \times m)$-matrix satisfying the system of PDEs

$$\xi_{a\mu}\partial_\mu A(x) + \eta_a A(x) = 0, \quad a = 1, \dots, N \tag{1.5.22}$$

and functions $\omega_{m+1}(x), \dots, \omega_{m+n-N}(x)$ form a complete set of functionally-independent first integrals of the system of PDEs (1.5.20).

We say that Ansatz (1.5.15) reduces system of PDEs (1.5.8) if the substitution of formulae (1.5.15) into (1.5.8) gives rise to a system of PDEs which is equivalent to one containing "new" independent ω_{m+1}, ω_{m+2}, …, ω_{m+n-N} and dependent φ^0, φ^1, …, φ^{m-1} variables only.

Let us recall the classical theorem about reduction of PDEs by means of group-invariant solutions: *a solution invariant under the N-dimensional Lie algebra with basis elements (1.5.11) satisfying the condition (1.5.10), which is a subalgebra of the symmetry algebra of PDE under study, reduces it to an $(n-N)$-dimensional PDE* [34, 190, 234, 235].

We will prove that for a given PDE to be reducible by means of the Ansatz (1.5.15) it is enough to require conditional invariance with respect to the corresponding involutive set of differential operators. Such a condition is essentially weaker than a requirement of invariance in the Lie sense and makes it possible to obtain principally new reductions of PDEs as compared with those obtainable within the framework of the classical Lie approach.

Definition 1.5.3. We say that the system of PDEs (1.5.8) is conditionally-invariant under the involutive set of differential operators (1.5.11) if the system of PDEs

$$\begin{cases} U_A(x, u, \underset{1}{u}, \dots, \underset{r}{u}) = 0, \quad A = 1, \dots, M, \\ \xi_{a\mu}(x,u)u^\alpha_{x\mu} - \eta^\alpha_a(x,u) = 0, \quad a = 1, \dots, N, \\ D\Big(\xi_{a\mu}(x,u)u^\alpha_{x\mu} - \eta^\alpha_a(x,u) = 0\Big), \quad a = 1, \dots, N, \\ \dots \\ D^{r-1}\Big(\xi_{a\mu}(x,u)u^\alpha_{x\mu} - \eta^\alpha_a(x,u) = 0\Big), \quad a = 1, \dots, N, \end{cases} \tag{1.5.23}$$

where the symbol $D^s(L = 0)$ denotes a set of all differential consequences of the equation $L = 0$ of order s, is invariant in the Lie sense under the one-parameter transformation groups having the generators Q_a, $a = 1, \dots, N$.

Before formulating the reduction theorem we will prove two auxiliary assertions.

Lemma 1.5.2. *Let us suppose that operators (1.5.11) form an involutive set. Then the set of differential operators*

$$Q'_a = \lambda_{ab}(x)Q_b, \quad \det \|\lambda_{ab}(x)\|_{a,b=1}^{N} \neq 0 \qquad (1.5.24)$$

is also involutive.

Proof. The lemma is proved by direct computation. Indeed,

$$\begin{aligned}[][Q'_a, Q'_b] &= [\lambda_{ac}Q_c, \lambda_{bd}Q_d] = \lambda_{ac}(Q_c\lambda_{bd})Q_d - \lambda_{bd}(Q_d\lambda_{ac})Q_c \\ &\quad +\lambda_{ac}\lambda_{bd}f_{cd}^{d_1}Q_{d_1} = \tilde{f}_{ab}^{c}Q_c = \tilde{f}_{ab}^{c}\lambda_{cd}^{-1}Q'_d.\end{aligned}$$

where λ_{cd}^{-1} are elements of the matrix inverse to the matrix $\|\lambda_{ab}(x)\|_{a,b=1}^{N}$. ▷

Lemma 1.5.3. *Let system of PDEs (1.5.8) be conditionally-invariant under the involutive set of differential operators (1.5.11). Then, it is conditionally-invariant under the involutive set (1.5.24) with arbitrary smooth functions λ_{ab}.*

Proof. To prove the lemma we need the following identity for coefficients of the s-th prolongation of the operator $\xi_\mu\partial_\mu + \eta^\alpha\partial_{u^\alpha}$:

$$\zeta^{\alpha}_{\mu_1\ldots\mu_i} = D_{\mu_1}\ldots D_{\mu_i}(\eta^\alpha - \xi_\mu u^\alpha_{x_\mu}) - \xi_\mu u^\alpha_{x_\mu x_{\mu_1}\ldots x_{\mu_i}}, \quad i = 1, 2, \ldots, s, \qquad (1.5.25)$$

where

$$D_\alpha = \partial_{x_\alpha} + u^\beta_{x_\alpha}\frac{\partial}{\partial u^\beta} + \sum_{n=1}^{\infty} u^\beta_{x_{\alpha_1}\ldots x_{\alpha_n}x_\alpha}\ \frac{\partial}{\partial(u^\beta_{x_{\alpha_1}\ldots x_{\alpha_n}})}$$

is a total differentiation operator with respect to the variable x_α. The above identity is proved by the method of mathematical induction. First, we will prove it under $i = 1$. From the prolongation formulae given in the introduction we have

$$\zeta^\alpha_\mu = D_\mu\eta^\alpha - u^\alpha_{x_\beta}D_\mu\xi_\beta = D_\mu(\eta^\alpha - \xi_\beta u^\alpha_{x_\beta}) - \xi_\beta u^\alpha_{x_\beta x_\mu},$$

whence it follows that the identity (1.5.25) holds for $i = 1$. Consequently, the base of induction is established.

Let us suppose now that the identity (1.5.25) holds for all $i \le k-1$. We will prove that hence its validity for $i = k$ follows.

Indeed,

$$\begin{aligned}\zeta^\alpha_{\mu_1\ldots\mu_k} &= D_{\mu_k}\zeta^\alpha_{\mu_1\ldots\mu_{k-1}} - u^\alpha_{x_{\mu_1}\ldots x_{\mu_{k-1}}x_\beta}D_{\mu_k}\xi_\beta \\ &= D_{\mu_k}\Big(D_{\mu_1}\ldots D_{\mu_{k-1}}(\eta^\alpha - \xi_\mu u^\alpha_{x_\mu}) - \xi_\mu u_{x_\mu x_{\mu_1}\ldots x_{\mu_{k-1}}}\Big) \\ &\quad - u^\alpha_{x_{\mu_1}\ldots x_{\mu_{k-1}}\partial x_\beta}D_{\mu_k}\xi_\beta = D_{\mu_1}\ldots D_{\mu_k}(\eta^\alpha - \xi_\mu u^\alpha_{x_\mu}) - \xi_\mu u^\alpha_{x_\mu x_{\mu_1}\ldots x_{\mu_k}},\end{aligned}$$

the same which is required.

Due to the identity proved above the r-th prolongation of the operator Q'_a being restricted to the set of solutions of system of PDEs (1.5.23) takes the form

$$\begin{aligned}\widetilde{Q}'_a &= Q'_a + D_{\mu_1}\dots D_{\mu_r}(\eta'^{\alpha}_a - \xi'_{a\mu}u^{\alpha}_{x_\mu})\frac{\partial}{\partial(u^{\alpha}_{x_{\mu_1}\dots\partial x_{\mu_r}})}\\ &\quad -\xi'_{a\mu}u^{\alpha}_{x_\mu x_{\mu_1}\dots x_{\mu_r}}\frac{\partial}{\partial(u^{\alpha}_{x_{\mu_1}\dots x_{\mu_r}})}.\end{aligned}$$

Substituting the formulae $\eta'^{\alpha}_a = \lambda_{ab}\eta^{\alpha}_b,\ \xi'_{a\mu} = \lambda_{ab}\xi_{b\mu}$ into the above equality and taking into account that the relations

$$D_{\mu_1}\dots D_{\mu_i}(\eta^{\alpha}_a - \xi_{a\mu}u^{\alpha}) = 0, \quad i = 1, 2, \dots, r-1$$

hold on the set of solutions of system (1.5.23) we get $\widetilde{Q}'_a = \lambda_{ab}(x,u)\widetilde{Q}_b$.

If we denote by the symbol L^i one of the equations of system (1.5.23) and by the symbol $[L]$ the set of its solutions, then the following equalities hold

$$\widetilde{Q}'_a L^i\Big|_{[L]} = \lambda_{ab}(x,u)\widetilde{Q}_b L^i\Big|_{[L]} = \lambda_{ab}(x,u)(\widetilde{Q}_b L^i\Big|_{[L]}) = 0,$$

whence it follows that the system of PDEs (1.5.8) is conditionally-invariant under the involutive set of operators (1.5.24). The lemma is proved. ▷

Theorem1.5.1. *Let the system of PDEs (1.5.8) be conditionally-invariant under the involutive set of differential operators (1.5.11) satisfying condition (1.5.10). Then, the Ansatz (1.5.15) invariant under the involutive set (1.5.11) reduces system of PDEs (1.5.8).*

Proof. Due to condition (1.5.10) there exists such a nonsingular $(N\times N)$-matrix $\|\lambda_{ab}(x,u)\|^N_{a,b=1}$ that

$$Q'_a = \lambda_{ab}(\xi_{b\mu}u^{\alpha}_{\mu} - \eta^{\alpha}_b) = u^{\alpha}_{x_{a-1}} + \sum_{\mu=N}^{n-1}\xi'_{a\mu}u^{\alpha}_{x_\mu} - \eta'^{\alpha}_a, \quad a = 1,\dots,N$$

and what is more the operators Q'_a form the involutive set (Lemma 1.5.1) such that system of PDEs (1.5.8) is conditionally-invariant with respect to it (Lemma 1.5.2).

Since the set of operators $Q'_a,\ a = 1,\dots,N$ is involutive, there exist such functions $f^c_{ab}(x,u)$ that

$$[Q'_a,\, Q'_b] = f^c_{ab}Q'_c, \quad a, b = 1,\dots,N. \tag{1.5.26}$$

Computing commutators on the left-hand sides of the above equalities and equating coefficients of the linearly independent differential operators ∂_{x_0}, ∂_{x_1}, $\ldots$, $\partial_{x_{N-1}}$ we have $f^c_{ab} = 0,\ a, b, c = 1, \ldots, N$. Consequently, operators Q'_a form a commutative Lie algebra.

Furthermore, systems of PDEs $Q_a\omega(x,u) = 0,\ a = 1, \ldots, N$ and $\lambda_{ab}(x,u) \times Q_b\omega(x,u) = 0,\ a = 1, \ldots, N$ with $\det \|\lambda_{ab}(x,u)\|^N_{a,b=1} \neq 0$ have the same set of functionally-independent first integrals. Hence we conclude that the involutive sets of operators Q_a and Q'_a give rise to the same Ansatz (1.5.15).

From the definition of the conditional invariance it follows that the system of PDEs

$$\begin{cases} U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = 1, \ldots, M, \\ u^\alpha_{x_{a-1}} + \sum_{\mu=N}^{n-1} \xi'_{a\mu} u^\alpha_{x_\mu} - \eta'^\alpha_a = 0, \quad a = 1, \ldots, N, \\ D\Big(u^\alpha_{x_{a-1}} + \sum_{\mu=N}^{n-1} \xi'_{a\mu} u^\alpha_{x_\mu} - \eta'^\alpha_a = 0\Big), \quad a = 1, \ldots, N, \\ \ldots \\ D^{r-1}\Big(u^\alpha_{x_{a-1}} + \sum_{\mu=N}^{n-1} \xi'_{a\mu} u^\alpha_{x_\mu} - \eta'^\alpha_a = 0\Big), \quad a = 1, \ldots, N \end{cases} \tag{1.5.27}$$

is invariant in Lie sense under the one-parameter groups generated by the mutually commuting operators Q'_a. Consequently, the above system is invariant in Lie sense under the commutative Lie algebra $\langle Q'_1, Q'_2, \ldots, Q'_N\rangle$.

Now we can apply the classical theorem about symmetry (group-theoretical) reduction of PDEs and conclude that the Ansatz invariant under the involutive set of operators (1.5.11) (or, which is the same, under the commutative Lie algebra $\langle Q'_1, Q'_2, \ldots, Q'_m\rangle$) reduces system of PDEs (1.5.27). But by construction all equations from the system (1.5.27) with the exception of the first m equations (which form the initial system of PDEs (1.5.8)) vanish identically on the manifold (1.5.11). Consequently, the Ansatz (1.5.11) reduces system (1.5.8). The theorem is proved. $\rhd$

Note 1.5.1. There exists a deep relation between reducibility of PDE (1.5.8) conditionally-invariant under the involutive set of operators (1.5.11) and compatibility of the over-determined system of PDEs (1.5.8), (1.5.9). But as is shown below from conditional invariance of PDE (1.5.8) with respect to the involutive set of operators (1.5.11) it does not follow a compatibility of system (1.5.8), (1.5.9) and *vise versa*.

The equation

$$(x_a x_a)(u_{x_b} u_{x_b}) - (x_a u_{x_a})^2 = m^2, \quad m \neq 0,$$

where $a, b = 1, 2, 3$, is invariant with respect to the rotation group $O(3)$ having the generators $J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\ a < b,\ a, b = 1, 2, 3$. However, the system

of PDEs

$$\begin{cases} (x_a x_a)(u_{x_b} u_{x_b}) - (x_a u_{x_a})^2 = m^2, \quad m \neq 0, \\ J_{ab} u = 0, \quad a, b = 1, 2, 3 \end{cases}$$

is incompatible, because substitution of the general solution of the last three equations $u = \varphi\,(x_a x_a)$ into the first one yields an inconsistent equality $0 = m^2$.

On the other hand, system of PDEs

$$\begin{cases} u_{xx} + u_{yy} - u + y(u_x - u) = 0, \\ u_y = 0 \end{cases}$$

is compatible (it has a solution $u = C\mathrm{e}^x,\ C = \mathrm{const}$) but the equation $u_{xx} + u_{yy} - u + y(u_x - u) = 0$ is not conditionally-invariant under the operator $Q = \partial_y$.

Note 1.5.2. We have proved Theorem 1.5.1 under assumption that the condition (1.5.10) holds. It is not difficult to prove that Theorem 1.5.1 is still valid, provided

$$\operatorname{rank} \|\xi_{a\mu}\|_{a=1}^{N}{}_{\mu=0}^{n-1} = \operatorname{rank} \|\xi_{a\mu}\, \eta_a^1, \ldots \eta_a^{m-1}\|_{a=1}^{N}{}_{\mu=0}^{n-1} = N' < N. \qquad (1.5.28)$$

Indeed, using transformation (1.5.24) we can reduce involutive set of operators (1.5.11) satisfying (1.5.28) to the form Q_1', …, $Q_{N'}$, $Q_{N'+1} = 0$, …, $Q_N = 0$. Now, we can apply Theorem 1.5.1 with $N = N'$. Consequently, if the system of PDEs (1.5.8) is conditionally-invariant under the involutive set of operators (1.5.11) satisfying (1.5.28), then the Ansatz (1.5.15) invariant under the involutive set (1.5.11) reduces it to $(n - N')$-dimensional PDE.

In the case when the condition (1.5.28) is not satisfied, so-called partially-invariant solutions (the term was introduced by Ovsjannikov [236]) are obtained. Reduction of PDEs conditionally-invariant under the involutive set of differential operators (1.5.11) not obeying the condition (1.5.28) is studied in detail in our paper [159].

2. Symmetry and compatibility of over-determined systems of linear PDEs. This subsection is devoted to the investigation of the following systems of PDEs:

$$B_{\mu\nu}(x)\partial_{x_\nu} + B_\mu(x)u(x) = 0, \quad \mu = 0, \ldots, n-1, \qquad (1.5.29)$$

where $x = (x_0,\, x_1, \ldots, x_{n-1}),\ u(x) = \Big(u^0(x),\, u^1(x),\, \ldots, u^{m-1}(x)\Big)^T,\ B_{\mu\nu},\ B_\mu$ are variable $(m \times m)$-matrices satisfying the condition

$$\operatorname{rank} \|B_{\mu\nu}(x)\|_{\mu,\nu=0}^{n-1} = n \times m. \qquad (1.5.30)$$

The problem of investigating compatibility of an over-determined system of the form (1.5.30) is closely connected with the problem of separation of variables in systems of linear PDEs (see [149, 227, 256] and Chapter 5).

Theorem 1.5.2. *System of PDEs (1.5.29) is compatible iff*

$$[B_{\mu\nu}\partial_\nu + B_\mu,\, B_{\alpha\beta}\partial_\beta + B_\alpha] = R_{\mu\alpha\beta}(B_{\beta\nu}\partial_\nu + B_\beta), \tag{1.5.31}$$

where $R_{\mu\alpha\beta}$ are some linear first-order differential operators with matrix coefficients, $\mu, \alpha = 0, \ldots, n-1$.

Proof. The necessity. Let system (1.5.30) be compatible. We will show that hence it follows that (1.5.31) holds. Due to (1.5.30) the block $(nm \times nm)$-matrix $\|B_{\mu\nu}\|_{\mu,\nu=0}^{n-1}$ is invertible. That is why there exists such a block $(nm \times nm)$-matrix $\|C_{\mu\nu}\|_{\mu,\nu=0}^{n-1}$ that

$$C_{\mu\nu}(x)B_{\nu\alpha}(x) = B_{\mu\nu}(x)C_{\nu\alpha}(x) = \delta_{\mu\alpha}I, \tag{1.5.32}$$

where I is the unit $(m \times m)$-matrix.

Let us rewrite (1.5.29) in the equivalent form

$$\partial_\mu u = F_\mu(x)u, \tag{1.5.33}$$

where $F_\mu = -C_{\mu\alpha}B_\alpha$.

It is well-known (see, for example, [43, 61, 261]) that the necessary and sufficient compatibility conditions of system of PDEs (1.5.33) read

$$\partial_\mu F_\nu - \partial_\nu F_\mu + [F_\mu, F_\nu] = 0, \quad \mu, \nu = 0, \ldots, n-1. \tag{1.5.34}$$

Introducing notations $Q_\mu = \partial_\mu - F_\mu(x)$ we rewrite (1.5.34) in the form

$$[Q_\mu,\, Q_\nu] = 0.$$

Representing the operators $B_{\mu\nu}\partial_\nu + B_\mu$ in the form $B_{\mu\nu}\partial_\nu + B_\mu = B_{\mu\nu}Q_\nu$ we compute the commutator

$$\begin{aligned}[B_{\mu\nu}Q_\nu,\, B_{\alpha\beta}Q_\beta] \;&=\; [B_{\mu\nu}, B_{\alpha\beta}]Q_\nu Q_\beta + B_{\mu\nu}[Q_\nu,\, B_{\alpha\beta}]Q_\beta \\ &\quad -B_{\alpha\beta}[Q_\nu,\, B_{\mu\beta}]Q_\beta.\end{aligned}$$

Finally, substituting formulae $Q_\mu = C_{\mu\alpha}(B_{\alpha\nu}\partial_\nu + B_\alpha)$ into the equality obtained we arrive at (1.5.31) and besides

$$R_{\mu\alpha\beta} = \{[B_{\mu\nu},\, B_{\alpha\beta_1}]Q_\nu + B_{\mu\nu}[Q_\nu,\, B_{\alpha\beta_1}] - B_{\alpha\nu}[Q_\nu,\, B_{\mu\beta_1}]\}C_{\beta_1\beta}.$$

The sufficiency. Given (1.5.30), we will prove that there exist such linear first-order operators $\widetilde{R}_{\mu\nu\beta}$ that the equality

$$[Q_\mu, Q_\nu] = \widetilde{R}_{\mu\nu\beta} Q_\beta \tag{1.5.35}$$

holds.

Indeed,

$$\begin{aligned}
[Q_\mu, Q_\alpha] \;\; = \;\; & [C_{\mu\alpha_1}(B_{\alpha_1\nu}\partial_\nu + B_{\alpha_1}), C_{\alpha\beta}(B_{\beta\nu_1}\partial_{\nu_1} + B_\beta)] \\
& = C_{\mu\alpha_1}[B_{\alpha_1\nu}\partial_\nu + B_{\alpha_1}, C_{\alpha\beta}](B_{\beta\nu_1}\partial_{\nu_1} + B_\beta) \\
& + C_{\mu\alpha_1}C_{\alpha\beta}[B_{\alpha_1\nu}\partial_\nu + B_{\alpha_1}, B_{\beta\nu_1}\partial_{\nu_1} + B_\beta] \\
& + [C_{\mu\alpha_1}, C_{\alpha\beta}](B_{\beta_1\nu}\partial_\nu + B_{\beta_1})(B_{\alpha\nu_1}\partial_{\nu_1} + B_\alpha) \\
& + C_{\alpha\beta}[C_{\mu\alpha_1}, B_{\beta\nu_1}\partial_{\nu_1} + B_\beta](B_{\alpha_1\nu}\partial_\nu + B_{\alpha_1}) \\
& = P_{\mu\alpha\beta}(B_{\beta\nu}\partial_\nu + B_\beta) = P_{\mu\alpha\beta}B_{\beta\nu}Q_\nu.
\end{aligned}$$

Choosing $P_{\mu\alpha\beta}B_{\beta\nu} = \widetilde{R}_{\mu\alpha\nu}$ we arrive at (1.5.35).

Computing the commutator on the left-hand side of (1.5.35) and equating coefficients of linearly independent operators ∂_μ we get the equalities

$$\widetilde{R}_{\mu\alpha\beta} = 0.$$

Consequently, operators Q_μ commute, i.e., conditions (1.5.34) hold identically. Hence it follows that system (1.5.33) is compatible. Since equations (1.5.33) are equivalent to the initial system of PDEs (1.5.29), the sufficiency is proved. $\triangleright$

Note 1.5.3. In the theory of non-Abelian gauge fields (Yang-Mills fields) conditions (1.5.34) are called the zero curvature equations. The general solution of the system of matrix PDEs (1.5.34) has the form

$$F_\mu = V_{x_\mu} V^{-1}, \quad \mu = 0, \ldots, 3, \tag{1.5.36}$$

where $V(x)$ is an arbitrary nonsingular $(m \times m)$-matrix which elements are smooth functions of x. Formula (1.5.36) establishes an one-to-one correspondence between over-determined systems (1.5.29) and solutions of the equation

$$B_{0\mu}(x)\partial_{x_\mu} + B_0(x)u(x) = 0 \tag{1.5.37}$$

of the form

$$u(x) = V(x)\chi,$$

where $V(x)$ is a nonsingular $(m \times m)$-matrix, $\chi = (\chi^0, \chi^1, \chi^2, \chi^3)^T$.

Thus, to construct particular solutions of the system of PDEs (1.5.37) it is necessary to classify algebraic objects of the type (1.5.29), (1.5.31). Up to now this problem is solved for a number of the Lie algebras and some simplest superalgebras [9, 10], [14]–[17], [100, 237, 238].

The most simple is the case where in (1.5.31) $R_{\mu\alpha\beta} = 0$ i.e., the operators $\Sigma_\mu = B_{\mu\nu}\partial_\nu + B_\mu$ commute. For many fundamental mathematical and theoretical physics equations (in particular, for the Dirac equation [198, 256]) it is possible to obtain complete description of commuting operators $\Sigma_\mu,\ \mu = 0, \dots, n-1$, where $\Sigma_0\psi = 0$ is the equation under investigation, and to construct solutions with separated variables. In this respect, we will consider the following particular case of system (1.5.29):

$$\Sigma_\mu u \equiv \Big(B_{\mu\nu}(x)\partial_\nu + B_\mu(x)\Big)u = \lambda_\mu u, \quad \mu = 0, \dots, n-1, \tag{1.5.38}$$

where $(\lambda_0, \lambda_1, \dots, \lambda_{n-1}) \in \Lambda \subset \mathbb{R}^n$, matrices $B_{\mu\nu}(x),\ B_\mu(x)$ being independent of λ_α.

When proving the principal assertion we will essentially use the following lemma.

Lemma 1.5.4. *If one of the systems of algebraic equations*

$$P_{\mu\alpha}B_{\mu\beta} + P_{\mu\beta}B_{\mu\alpha} = 0, \tag{1.5.39}$$

$$P_{\alpha\mu}C_{\mu\beta} + P_{\beta\mu}C_{\mu\alpha} = 0, \tag{1.5.40}$$

where $\|B_{\mu\nu}(x)\|_{\mu,\nu=0}^{n-1}$ *is a nonsingular block* $(nm \times nm)$*-matrix,* $\|C_{\mu\nu}(x)\|_{\mu,\nu=0}^{n-1}$ *is its inverse and* $P_{\mu\alpha}$ *are some variable* $(m \times m)$*-matrices, holds true, then* $P_{\mu\alpha} = 0$.

Proof. We prove the lemma under assumption that (1.5.39) holds. Let us rewrite equalities (1.5.39) in the equivalent form

$$P_{\mu\mu_1}C_{\mu_1\nu_1}T_{\nu_1\mu\alpha\beta} = 0. \tag{1.5.41}$$

Here $T_{\nu_1\mu\alpha\beta} = B_{\nu_1\alpha}B_{\mu\beta} + B_{\nu_1\beta}B_{\mu\alpha}$.

Since the identities

$$C_{\mu\nu}C_{\mu_1\nu_1}T_{\nu_1\nu\alpha\beta} = (\delta_{\mu_1\alpha}\delta_{\mu\beta} + \delta_{\mu_1\beta}\delta_{\mu\alpha})I$$

hold, the block matrix $\|T_{\nu_1\mu\alpha\beta}\|$ is invertible. Consequently, equation (1.5.41) is equivalent to the relation

$$P_{\mu\alpha}C_{\alpha\nu} = 0. \tag{1.5.42}$$

Multiplying (1.5.42) by $B_{\nu\beta}$ and summing over ν we have

$$P_{\mu\beta} = 0, \quad \mu, \beta = 0, \ldots, n-1.$$

In the case, where (1.5.40) holds, the proof is analogous. $\rhd$

Theorem 1.5.3. *Provided (1.5.30) holds the system of PDEs (1.5.38) is compatible iff*

$$[B_{\mu\nu}\partial_\nu + B_\mu,\, B_{\alpha\beta}\partial_\beta + B_\alpha] = 0, \quad \mu, \alpha = 0, \ldots, n-1. \tag{1.5.43}$$

Proof. According to Theorem 1.5.2, the compatibility criterion for the system of PDEs (1.5.38) reads

$$\begin{aligned} &[B_{\mu\nu}\partial_\nu + B_\mu - \lambda_\mu,\, B_{\alpha\beta}\partial_\beta + B_\alpha - \lambda_\alpha] \\ &\quad = (R_{\mu\alpha\beta\nu}\partial_\nu + R_{\mu\alpha\beta})(B_{\beta\nu_1}\partial_{\nu_1} + B_\beta - \lambda_\beta). \end{aligned} \tag{1.5.44}$$

Computing the commutator in the left-hand side and equating coefficients of the linearly independent operators $\partial_\nu\partial_\beta,\ \partial_\beta,\ I$ we get the system of PDEs for matrix functions $B_{\mu\nu},\ B_\mu,\ R_{\mu\alpha\beta\nu},\ R_{\mu\alpha\beta}$

$$[B_{\mu\nu},\, B_{\alpha\beta}] + [B_{\mu\beta},\, B_{\alpha\nu}] = R_{\mu\alpha\mu_1\nu}B_{\mu_1\beta} + R_{\mu\alpha\mu_1\beta}B_{\mu_1\nu}, \tag{1.5.45}$$

$$\begin{aligned} &B_{\mu\nu}\partial_\nu B_{\alpha\beta} - B_{\alpha\nu}\partial_\nu B_{\mu\beta} + [B_{\mu\beta},\, B_\alpha] - [B_{\alpha\beta},\, B_\mu] \\ &\quad = R_{\mu\alpha\mu_1\nu}\partial_\nu B_{\mu_1\beta} + R_{\mu\alpha\mu_1\beta}(B_{\mu_1} - \lambda_{\mu_1}) + R_{\mu\alpha\mu_1}B_{\mu_1\beta}, \end{aligned} \tag{1.5.46}$$

$$\begin{aligned} &B_{\mu\nu}\partial_\nu B_\alpha - B_{\alpha\nu}\partial_\nu B_\mu + [B_\mu,\, B_\nu] = R_{\mu\alpha\mu_1\nu}\partial_\nu B_{\mu_1} \\ &\quad + R_{\mu\alpha\mu_1}(B_{\mu_1} - \lambda_{\mu_1}). \end{aligned} \tag{1.5.47}$$

Differentiating (1.5.45) with respect to λ_{α_1} we arrive at the relations

$$\frac{\partial R_{\mu\alpha\mu_1\nu}}{\partial\lambda_{\alpha_1}}B_{\mu_1\beta} + \frac{\partial R_{\mu\alpha\mu_1\beta}}{\partial\lambda_{\alpha_1}}B_{\mu_1\nu} = 0,$$

whence due to Lemma 1.5.4 it follows that

$$\frac{\partial R_{\mu\alpha\beta\nu}}{\partial\lambda_{\mu_1}} = 0, \quad \mu, \alpha, \beta, \nu, \mu_1 = 0, \ldots, n-1.$$

Differentiation of equality (1.5.46) with respect to λ_{α_1} yields

$$\frac{\partial R_{\mu\alpha\mu_1}}{\partial\lambda_{\alpha_1}}B_{\mu_1\beta} - R_{\mu\alpha\alpha_1\beta} = 0.$$

Multiplying the above equality by $C_{\beta\beta_1}$ and summing over β we have

$$\frac{\partial R_{\mu\alpha\beta_1}}{\partial\lambda_{\alpha_1}} = R_{\mu\alpha\alpha_1\beta} C_{\beta\beta_1}$$

or

$$R_{\mu\alpha\beta_1} = \lambda_{\alpha_1} R_{\mu\alpha\alpha_1\beta} C_{\beta\beta_1} + \widetilde{R}_{\mu\alpha\beta_1}, \qquad (1.5.48)$$

and besides $\partial\widetilde{R}_{\mu\alpha\beta_1}/\partial\lambda_{\alpha_1} = 0,\ \alpha_1 = 0, \ldots, n-1$.

Substituting (1.5.48) into (1.5.47) and equating coefficients of λ_{α_1}, $\lambda_{\mu_1}\lambda_{\alpha_1}$ we come to the following relations:

$$R_{\mu\alpha\alpha_1\beta} C_{\beta\beta_1} + R_{\mu\alpha\beta_1\beta} C_{\beta\alpha_1} = 0,$$

$$R_{\mu\alpha\alpha_1\beta} C_{\beta\beta_1} B_{\beta_1} - \widetilde{R}_{\mu\alpha\alpha_1} = 0. \qquad (1.5.49)$$

According to Lemma 1.5.4 $R_{\mu\alpha\alpha_1\beta} = 0$, whence it follows that $\widetilde{R}_{\mu\alpha\alpha_1} = 0$.

Thus, the necessary and sufficient compatibility conditions for system (1.5.38) are given by relations (1.5.44) with $R_{\mu\alpha\beta\nu} = R_{\mu\alpha\beta} = 0$ or, which is the same, by relations (1.5.43). The theorem is proved. $\triangleright$

Results obtained in the present section are applied in a sequel to reduce multi-dimensional nonlinear partial differential equations to ODEs and to construct their exact solutions in explicit form. In addition, Theorems 1.5.2, 1.5.3 form a basis of our approach to separation of variables in systems of linear PDEs (see Chapter 5).

1.6. Conservation laws

One of the important properties of equations admitting a nontrivial symmetry group is the existence of constants of motion (by a constant of motion we mean some combination of solutions of the equation considered which preserves its value in time). The well-known examples of constants of motion are the energy, the momentum and the angular momentum.

Within the framework of the traditional approach to the problem of construction of constants of motion, going back to Noether's works, we have to investigate symmetry of the Lagrangian of the equation in question and to construct conservation laws with the help of the Noether theorem [35, 190]. This theorem establishes correspondence between one-parameter subgroups of the symmetry group of the Lagrangian and conservation laws. However the above

approach has restricted applicability since there exist mathematical physics equations which can not be derived via the Lagrange function. In addition, there are examples of conservation laws which cannot be obtained with the help of the Noether theorem even for equations derived in the framework of the variational principle [115, 116, 118, 190, 280].

That is why we apply a method of construction of constants of motion for the Dirac equation based on the direct calculation of a conserved quantity as a zero component of the four-vector of current with components $j_\mu = j_\mu(x, \bar{\psi}, \psi, \underset{1}{\bar{\psi}}, \underset{1}{\psi}, \ldots)$, $\mu = 0, \ldots, 3$ which satisfies the continuity condition

$$\partial_\mu j_\mu = 0 \tag{1.6.1}$$

for each solution $\psi = \psi(x)$ of the Dirac equation.

Lemma 1.6.1. *Let us suppose that there exists the four-vector of current satisfying the relation (1.6.1) and besides conditions*

$$j_a \to 0, \quad a = 1, 2, 3 \quad \textit{under} \quad |\vec{x}| \to +\infty$$

hold true. Then, the quantity

$$I = \int\limits_{\mathbb{R}^3} j_0 \, d^3x \tag{1.6.2}$$

is conserved in time, i.e., $\partial I/\partial x_0 = 0$.

Proof. Differentiating (1.6.2) with respect to x_0 yields

$$\partial I/\partial x_0 = \int\limits_{\mathbb{R}^3} (\partial j_0/\partial x_0) \, d^3x,$$

whence it follows

$$\partial I/\partial x_0 = -\int\limits_{\mathbb{R}^3} (\partial_a j_a) \, d^3x.$$

Applying the Gauss-Ostrogradski theorem we get $\partial I/\partial x_0 = 0$. The lemma is proved. $\rhd$

For brevity we will call the four-vector of current satisfying relation (1.6.1) on the set of solutions of the Dirac equation the conservation law.

Up to now there is no effective algorithm making it possible to obtain all conservation laws admitted by an arbitrary PDE. We will construct conservation laws for the Dirac equation following an approach suggested in [115, 116] which utilizes its Lie and non-Lie symmetry.

Lemma 1.6.2. *Let Q be a symmetry operator of the Dirac equation (1.1.1). Then, the four-vector with components*

$$j_\mu = \bar{\psi}\gamma_\mu Q\psi, \tag{1.6.3}$$

where $\psi = \psi(x)$ is an arbitrary solution of PDE (1.1.1) vanishing under $|\vec{x}| \to +\infty$, is a conservation law.

The proof is carried out by direct verification

$$\begin{aligned}\partial_\mu j_\mu \quad &= \quad \partial_\mu(\bar{\psi}\gamma_\mu Q\psi) = (\partial_\mu\bar{\psi}\gamma_\mu)Q\psi + \bar{\psi}\gamma_\mu\partial_\mu Q\psi = im\bar{\psi}Q\psi \\ &\quad\; -im\bar{\psi}Q\psi = 0\end{aligned}$$

(we use the fact that any symmetry operator Q transforms the set of solutions of PDE (1.1.1) into itself, i.e., $i\gamma_\mu\partial_\mu Q\psi = mQ\psi$). $\rhd$

Let us find explicit expressions of conservation laws corresponding to the symmetry operators of the Dirac equation which belong to the class $\mathcal{M}_1$ (see Section 1.1). Substituting the basis elements of the Poincaré algebra $AP(1,3)$ $P_\mu,\ J_{\mu\nu}$ into (1.6.3) we get the well-known expressions of the energy-momentum and angular-momentum tensors

$$T_{\mu\nu} = \bar{\psi}\gamma_\mu\partial_\mu\psi, \quad \Omega_{\mu\alpha\beta} = \bar{\psi}\gamma_\mu J_{\alpha\beta}\psi \tag{1.6.4}$$

satisfying continuity equation (1.6.1) on the index μ.

A trivial identity operator I gives rise to the current of a probability density

$$T_\mu = \bar{\psi}\gamma_\mu\psi. \tag{1.6.5}$$

Substitution of zero components of currents (1.6.4), (1.6.5) into formula (1.6.2) yields the following conserved quantities:

a) the energy

$$E = \int\limits_{\mathbb{R}^3} \psi^\dagger\partial_0\psi\, d^3x;$$

b) the momentum

$$P_a = \int\limits_{\mathbb{R}^3} \psi^\dagger\partial_a\psi\, d^3x;$$

c) the angular momentum

$$\omega_{\alpha\beta} = \int\limits_{\mathbb{R}^3} \psi^\dagger\Big(x_\alpha\partial^\beta - x_\beta\partial^\alpha - (1/2)\gamma_\alpha\gamma_\beta\Big)\psi\, d^3x, \quad \alpha \neq \beta;$$

d) the probability

$$p = \int\limits_{\mathbb{R}^3} \psi^\dagger \psi \, d^3x.$$

Constants of motion corresponding to the non-Lie symmetry operators of the Dirac equation (1.1.34) are obtained in the same way.

In the case of the massless Dirac equation (1.1.17) there arise additional conserved quantities (for more detail see [118]). We restrict ourselves to adducing constants of motion which correspond to symmetry operators of equation (1.1.17) not belonging to an enveloping algebra of the conformal algebra $AC(1,3)$

$$I^1_{\mu\nu} = \int\limits_{\mathbb{R}^3} \psi^\dagger (\gamma_\mu \partial^\nu - \gamma_\nu \partial^\mu) \psi \, d^3x, \quad I^1_\mu = \int\limits_{\mathbb{R}^3} \psi^\dagger A_\mu \psi d^3x,$$

$$I^2_{\mu\nu} = \int\limits_{\mathbb{R}^3} \psi^\dagger \Big([K_\mu, \, A_\nu] - [K_\nu, \, A_\mu] \Big) \psi d^3x, \quad I^2_\mu = \int\limits_{\mathbb{R}^3} \psi^\dagger \gamma_4 A_\mu \psi d^3x,$$

where $A_\mu = \gamma_\mu x_\nu \partial_\nu - x^\nu \gamma_\nu \partial^\mu - 2\gamma_\mu, \ \mu, \nu = 0, \ldots, 3.$

EXACT SOLUTIONS

The present chapter is devoted to exact solutions of Poincaré-invariant systems of nonlinear PDEs for spinor, vector and scalar fields. We establish the necessary and sufficient compatibility conditions and construct the general solution of the system of nonlinear PDEs which consists of the nonlinear d'Alembert and Hamilton equations. With the use of subgroup structure of the groups $P(1,3), \widetilde{P}(1,3), C(1,3)$ we construct Ansätze reducing multi-dimensional spinor and vector equations to PDEs of lower dimension. These Ansätze enable us to obtain multi-parameter families of exact solutions of the nonlinear Dirac, Maxwell-Dirac and Dirac-d'Alembert equations, some of the families containing arbitrary functions. In particular, the exact solutions of the nonlinear Dirac equation expressed via the Bessel, Weierstrass, Gauss and Chebyshev-Hermite functions are constructed. In addition, a method of constructing exact solutions of PDEs for scalar, vector and tensor fields via solutions of a nonlinear spinor equation is suggested.

2.1. On compatibility and general solution of the d'Alembert–Hamilton system

As shown in [123, 156, 165] the substitution

$$w(x) = \varphi\Big(u(x)\Big), \quad \varphi \in C^2(\mathbb{R}^1, \mathbb{R}^1) \tag{2.1.1}$$

reduces the n-dimensional nonlinear d'Alembert equation

$$\Box_n w \equiv \frac{\partial^2 w}{\partial x_0^2} - \triangle_{n-1} w = F_0(w) \tag{2.1.2}$$

to ODE for a function $\varphi(u)$ iff the scalar function $u = u(x_0, x_1, \ldots, x_{n-1})$ satisfies the nonlinear d'Alembert and Hamilton equations

$$\Box_n u = F_1(u), \qquad (2.1.3)$$

$$(\partial_A u)(\partial^A u) = F_2(u), \qquad (2.1.4)$$

simultaneously.

In the above formulae F_1, F_2 are arbitrary smooth functions depending on u only. Hereafter in the present section we suppose that indices denoted by A, B, C take the values $0, \ldots, n-1$ and besides the summation convention in the pseudo-Euclidean space $M(1, n-1)$ with the metric tensor $g_{AB} = \mathrm{diag}\,(1, -1, \ldots, -1)$ is implied.

Thus, to obtain all Ansätze of the form (2.1.1) reducing equation (2.1.2) to an ODE one has to construct the general solution of system (2.1.3), (2.1.4). Let us emphasize that such an approach to the problem of reduction of equation (2.1.2) does not require the knowledge of a subgroup structure of the invariance group.

Following [154, 156] we call the system of PDEs (2.1.3), (2.1.4) the d'Alembert-Hamilton system.

The d'Alembert-Hamilton system plays an important role in the theory of Poincaré-invariant equations for the scalar [137, 154, 156, 171], spinor [151, 155] and vector fields. In particular, any second-order $P(1, n-1)$-invariant scalar equation can be reduced to ODE with the use of solutions of system (2.1.3), (2.1.4) (without applying the symmetry reduction technique).

The three-dimensional elliptic analogue of system of PDEs (2.1.3), (2.1.4)

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = 0, \quad u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2 = 0$$

with a complex-valued function $u(\vec{x})$ was studied by Jacobi [25], who constructed the following class of its exact solutions

$$C_0(u) + C_1(u)x_1 + C_2(u)x_2 + C_3(u)x_3 = 0, \qquad (2.1.5)$$

where $C_0(u), \ldots, C_3(u)$ are arbitrary smooth complex-valued functions satisfying the equality

$$C_1(u)^2 + C_2(u)^2 + C_3(u)^2 = 0. \qquad (2.1.6)$$

Later on, Smirnov and Sobolev [263, 264] proved that the formulae (2.1.5), (2.1.6) give the general solution of the above over-determined system of PDEs. Some classes of exact solutions of the system of PDEs (2.1.3), (2.1.4) were obtained by Bateman [27], Cartan [44] and Erugin [77].

Recently, Collins [55] has obtained the general solution of the three-dimensional d'Alembert-Hamilton system using the methods of differential geometry. However approach cannot be applied to systems of PDEs (2.1.3), (2.1.4) having $n > 3$ independent variables.

In the present section we will establish the necessary compatibility conditions of system (2.1.3), (2.1.4) for arbitrary $n \in \mathbb{N}$ and obtain its compatibility criterion in the case $n = 4$. Next, we will construct the general solution of the four-dimensional d'Alembert-Hamilton system.

1. Compatibility of over-determined system of PDEs (2.1.3), (2.1.4). We study the matter of compatibility of the d'Alembert-Hamilton system under assumption that $u(x)$ is a complex-valued function of n complex variables $x_0,\ x_1, \ldots, x_{n-1}$. Provided $F_2(u) \neq 0$, we can transform system (2.1.3), (2.1.4) by means of changing the dependent variable

$$u \to u' = \int^{u} \Big(F_2(\tau)\Big)^{-1/2} d\tau \tag{2.1.7}$$

as follows

$$\Box_n u' = F(u'), \quad (\partial_A u')(\partial^A u') = 1.$$

Consequently, the problem of investigating compatibility of the d'Alembert-Hamilton system is reduced to studying compatibility of the system of PDEs

$$\Box_n u = F(u), \quad (\partial_A u)(\partial^A u) = \lambda, \tag{2.1.8}$$

where λ is a discrete parameter taking the values $0,\ 1$.

To solve the above problem we will need the following auxilliary results.

Lemma 2.1.1[171]. *Solutions of the system (2.1.8) satisfy the identities*

$$\begin{aligned}
&u_{AB}u^{AB} = -\lambda \dot{F}(u),\\
&u_{AB_1}u^{B_1B_2}u^{A}_{B_2} = \frac{\lambda^2}{2!}\ddot{F}(u),\\
&\ldots,\\
&u_{AB_1}u^{B_1B_2}\cdot\ldots\cdot u^{B_mA} = \frac{(-\lambda)^m}{m!}F^{(m)}(u), \quad m \geq 1,
\end{aligned} \tag{2.1.9}$$

where $u_{AB} = \partial_A\partial_B u,\ u^B_A = g_{BC}u_{CA},\ A, B, C = 0, \ldots, n-1,\ F^{(m)} = d^m F/du^m$.

Proof. We prove the assertion by means of the mathematical induction method by m. Differentiating the second equation of system (2.1.8) with

respect to $x_B,\ x_C$ we have

$$u_{ABC}u^A + u_{AB}u^A_C = 0. \tag{2.1.10}$$

Convoluting (2.1.10) with the metric tensor g_{AB} we arrive at the equality

$$u_{AB}u^{AB} + u^A \Box_n u_A = 0.$$

Since $\Box_n u_A = \partial_A F(u) = u_A \dot{F}(u)$, the above expression is rewritten in the form

$$u_{AB}u^{AB} + \lambda \dot{F}(u) = 0.$$

Consequently, the base of induction is ensured. Let us assume that the assertion holds for $m = k \in \mathbb{N}$. We will prove that it holds for $m = k+1$ as well.

Convoluting (2.1.10) with the tensor

$$u^{BB_2}u_{B_2B_3}\cdot\cdots\cdot u^{B_kC},$$

gives

$$\begin{gathered} u_{AB}u^{BB_2}u_{B_2B_3}\cdot\cdots\cdot u^{B_kC}u^A_C \\ +u^A u_{ABC}u^{BB_2}u_{B_2B_3}\cdot\cdots\cdot u^{B_kC} = 0. \end{gathered} \tag{2.1.11}$$

Since, according to the assumption of the induction, the equalities

$$\begin{aligned} & u^A u_{ABC}u^{BB_2}u_{B_2B_3}\cdot\cdots\cdot u^{B_kC} = (k+1)^{-1}u^A\partial_A \\ & \quad\times\Big(u_{BC}u^{BB_2}u_{B_2B_3}\cdot\cdots\cdot u^{B_kC}\Big) = (k+1)^{-1}u^A\partial_A \\ & \quad\times(k!)^{-1}(-\lambda)^k F^{(k)}(u) = -\Big((k+1)!\Big)^{-1}(-\lambda)^{k+1}F^{(k+1)}(u) \end{aligned}$$

hold, from (2.1.11) it follows that

$$u_{AB_1}u^{B_1B_2}\cdot\cdots\cdot u^{B_{k+1}A} = \Big((k+1)!\Big)^{-1}(-\lambda)^{k+1}F^{(k+1)}(u).$$

The lemma is proved. $\triangleright$

Lemma 2.1.2[171]. *Solutions of the system of PDEs (2.1.8) satisfy the n-dimensional Monge-Ampère equation*

$$\det \|u_{x_Ax_B}\|^{n-1}_{A,B=0} = 0. \tag{2.1.12}$$

Proof. The assertion follows from the fact that (2.1.12) is a criterion of functional dependence of functions $u_{x_0},\ u_{x_1},\ldots,u_{x_{n-1}}$. $\triangleright$

Theorem 2.1.1. *Let the d'Alembert-Hamilton system (2.1.8) be compatible. Then*

$$F(u) = \lambda \dot{f}(u) f^{-1}(u), \tag{2.1.13}$$

and what is more

$$\frac{d^n f(u)}{du^n} = 0. \tag{2.1.14}$$

Proof. The cases $\lambda = 1$ and $\lambda = 0$ have to be considered separately.

The case $\lambda = 1$. Due to the Hamilton-Cayley theorem [173] an arbitrary $(n \times n)$-matrix $W = \|W_{AB}\|_{A,B=0}^{n-1}$ satisfies the following identity:

$$\sum_{k=0}^{n-1} (-1)^k \Sigma(M_k)\ \mathrm{tr}\,(W^{n-k}) + (-1)^n n\ \det W = 0, \tag{2.1.15}$$

where $\mathrm{tr}\,\|W_{AB}\|_{A,B=0}^{n-1} = \sum\limits_{C=0}^{n-1} W_{CC}$ is the trace of a matrix W.

In (2.1.15) we designate the sum of k-th order principal minors of the matrix M by the symbol $\Sigma(M_k)$. This sum is determined by the recurrent formula

$$\begin{aligned} &\Sigma(M_k) = k^{-1}(-1)^{k-1}\left\{\sum_{l=0}^{k-1}(-1)^l \Sigma(M_l)\ \mathrm{tr}\,(W^{k-l})\right\}, \quad k \geq 1, \\ &\Sigma(M_0) \stackrel{\mathrm{def}}{=} 1. \end{aligned} \tag{2.1.16}$$

We choose the matrix elements W_{AB} as follows

$$W_{AB} = \partial_A \partial^B u(x), \quad A, B = 0, \ldots, n-1,$$

whence due to Lemmas 2.1.1, 2.1.2 we conclude that

$$\mathrm{tr}\,(W^k) = \frac{1}{(k-1)!} F^{(k-1)}, \quad \det W = 0. \tag{2.1.17}$$

Substitution of the above formulae into (2.1.15) gives rise to an ODE for $F(u)$. Let us prove that this ODE is transformed to the form (2.1.14) by means of a nonlocal change of the dependent variable (2.1.13).

Introducing the notation

$$Y_N = \sum_{k=0}^{N} (-1)^k \Sigma(M_k) \mathrm{tr}\,(W^{N-k+1}),$$

we rewrite formula (2.1.16) as follows

$$\Sigma(M_k) = \frac{(-1)^{k-1}}{k} Y_{k-1}, \quad k \geq 1,$$

whence

$$\begin{aligned} Y_N &= \mathrm{tr}\,(W^{N+1}) - \sum_{k=1}^{N} \frac{1}{k} Y_{k-1} \mathrm{tr}\,(W^{N-k+1}) = \frac{(-1)^N}{N!} \left(\frac{\dot f}{f}\right)^{(N)} \\ &+ \sum_{k=1}^{N} \frac{(-1)^{N-k-1}}{k(n-k)!} Y_{k-1} \left(\frac{\dot f}{f}\right)^{(N-k)}, \quad N \geq 1, \\ Y_0 &= \frac{\dot f}{f}. \end{aligned} \tag{2.1.18}$$

Using the mathematical induction method we will prove the equalities

$$Y_N = \frac{(-1)^N}{N!} \frac{f^{(N+1)}}{f}, \quad N \geq 1. \tag{2.1.19}$$

Let us prove that (2.1.19) holds under $N = 1$. Due to (2.1.17) an expression for Y_1 can be rewritten in the following way:

$$Y_1 = \mathrm{tr}\,(W^2) - \Sigma(M_1)\mathrm{tr}\,W = -\dot F - F^2.$$

Substitution of $F = \dot f / f$ into the above equality yields $Y_1 = -\dddot f / f$. The base of induction is established.

Let us assume that (2.1.19) holds for all $m \leq N - 1$. We will prove that (2.1.19) holds for $m = N$ as well.

Indeed,

$$\begin{aligned} Y_N \quad &= \quad \frac{(-1)^N}{N!} \left(\frac{\dot f}{f}\right)^{(N)} + \sum_{k=1}^{N} \frac{(-1)^{N-k-1}}{k(N-k)!} \left(\frac{\dot f}{f}\right)^{(N-k)} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{f^{(k)}}{f}\right) \\ &= \frac{(-1)^N}{N!} \left(\frac{\dot f}{f}\right)^{(N)} + \sum_{k=1}^{N} \frac{(-1)^N}{k!(N-k)!} \left(\frac{\dot f}{f}\right)^{(N-k)} \left(\frac{f^{(k)}}{f}\right) \\ &= \frac{(-1)^N}{N! f} \sum_{k=0}^{N} \mathrm{C}_N^k \left(\frac{\dot f}{f}\right)^{(N-k)} f^{(k)} = \frac{(-1)^N}{N! f} \frac{f^{(N+1)}}{f}. \end{aligned}$$

Consequently, relation (2.1.19) holds for all $N \in \mathbb{N}$. Putting $N = n - 1$ yields

$$Y_{n-1} = \frac{(-1)^{n-1}}{(n-1)!} \frac{f^{(n)}}{f}.$$

On the other hand, using (2.1.15), (2.1.17) we come to the following relation:

$$Y_{n-1} = (-1)^{n+1} n \det W = 0,$$

whence $f^{(n)}(u) = 0$.

The case $\lambda = 0$. Taking into account Lemmas 2.1.1, 2.1.2 yields

$$\det W = 0, \quad \operatorname{tr}(W^k) = 0, \quad k = 2, \ldots, n-1.$$

Due to these equalities formulae (2.1.15), (2.1.16) take the form

$$(-1)^{n-1} F \Sigma(M_{n-1}) = 0, \tag{2.1.20}$$

$$\Sigma(M_0) = 1, \quad \Sigma(M_k) = \frac{F}{k} \Sigma(M_{k-1}). \tag{2.1.21}$$

Resolving the recurrent relations (2.1.21) with respect to $\Sigma(M_k)$ we get

$$\Sigma(M_k) = \frac{F^k}{k!}, \quad k \geq 1.$$

Inserting $\Sigma(M_{n-1}) = \Big((n-1)!\Big)^{-1} F^{n-1}$ into (2.1.20) we have

$$\frac{(-1)^{n-1}}{(n-1)!} F^n = 0,$$

whence $F = 0$. The theorem is proved. $\rhd$

Consequence 2.1.1. *The over-determined system of PDEs*

$$\Box_n u = F(u), \quad (\partial_A u)(\partial^A u) = 0 \tag{2.1.22}$$

is compatible iff $F(u) \equiv 0$.

Proof. The necessity is a direct consequence of Theorem 2.1.1. To prove the sufficiency we will show that system (2.1.19) with $F(u) = 0$ possesses nontrivial solutions. It is straightforward to check that the function $u(x) = C_1(x_0 + x_3) + C_2$, where $C_1,\ C_2$ are constants, satisfies equations (2.1.19) under $F(u) = 0$, the same as what was to be proved. $\rhd$

Let us note that the compatibility criterion for the system of PDEs (2.1.19) with a real-valued function $u = u(x)$ has been established in [51].

Let us say a few words about geometrical interpretation of the d'Alembert-Hamilton system. If we designate by $P_k(u)$ a k-th order polynomial, then the

necessary compatibility conditions (2.1.13), (2.1.14) can be represented in the form

$$F(u) = \lambda \frac{d}{du} \ln P_k(u), \quad 0 \le k \le n-1.$$

Let $\alpha_1,\ \alpha_2, \ldots, \alpha_k$ be the roots of the polynomial $P_k(u)$. Then, the above relations read

$$F(u) = \lambda \sum_{i=1}^{k} \frac{1}{u+\alpha_i}, \quad 1 \le k \le n-1 \tag{2.1.23}$$

or

$$F(u) = 0, \quad k = 0. \tag{2.1.24}$$

According to [51, 165, 270] the parameters

$$\begin{aligned} \varkappa_i &= (\alpha_i)^{-1}, \quad i = 1, \ldots, k, \\ \widetilde{\varkappa}_j &= 0, \quad j = k+1, \ldots, n-1 \end{aligned}$$

can be interpreted as the principal curvatures of the level surface of the solution of system (2.1.8), (2.1.23) under $\lambda = 1$. Consequently, solutions of the d'Alembert-Hamilton system have the remarkable geometrical property: their level surfaces have all principal curvatures constant (for the first time this fact was established by Cartan [44]).

Now we adduce an assertion giving the compatibility criterion of the nonlinear d'Alembert-Hamilton system (2.1.3), (2.1.4) in the case $n = 4$

$$\Box u = F_1(u), \quad (\partial_\mu u)(\partial^\mu u) = F_2(u). \tag{2.1.25}$$

Here $u = u(x_0, x_1, x_2, x_3) \in C^2(\mathbb{C}^4,\ \mathbb{C}^1),\ \ \{F_1, F_2\} \subset C(\mathbb{C}^1,\ \mathbb{C}^1)$.

Theorem 2.1.2. *System of PDEs (2.1.25) is compatible iff the functions* $F_1,\ F_2$ *have the form*

$$\begin{aligned} &1)\quad F_1(u) = F_2(u) = 0, \ \ \textit{or} \\ &2)\quad F_1(u) = N(\dot{f} f)^{-1} - \ddot{f}(\dot{f})^{-3}, \quad F_2(u) = (\dot{f})^{-2}, \end{aligned} \tag{2.1.26}$$

where $f = f(u) \in C^2(\mathbb{C}^1,\ \mathbb{C}^1)$ *is an arbitrary function satisfying the condition* $\dot{f} \not\equiv 0$, N *is a discrete parameter taking the values* $0, 1, 2, 3$.

The proof can be found in [165].

Note 2.1.1. System of PDEs (2.1.25) with $F_1,\ F_2$ given by formulae (2.1.26) is transformed to the form

$$\Box u = N u^{-1}, \quad (\partial_\mu u)(\partial^\mu u) = 1 \tag{2.1.27}$$

by means of the change of the dependent variable

$$u \to u' = f(u). \tag{2.1.28}$$

Theorem 2.1.3. *Let $u = u(x)$ be a real-valued function of four real variables $x_0,\ x_1,\ x_2,\ x_3$. Then, system (2.1.25) is compatible iff the functions $F_1,\ F_2$ have the form*

$$\begin{aligned} &1)\quad F_1(u) = F_2(u) = 0, \ \ \textit{or}\\ &2)\quad F_1(u) = \varepsilon N(\dot{f} f)^{-1} - \varepsilon \ddot{f}(\dot{f})^{-3}, \quad F_2(u) = \varepsilon(\dot{f})^{-2}, \end{aligned} \tag{2.1.29}$$

where $f = f(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function satisfying the condition $\dot{f} \not\equiv 0$; N is a discrete parameter taking the values $0, 1, 2, 3$; $\varepsilon = \pm 1$.

Note 2.1.2. System of PDEs (2.1.25) with $F_1,\ F_2$ given by formulae (2.1.29) is transformed to the form

$$\Box u = \varepsilon N u^{-1}, \quad (\partial_\mu u)(\partial^\mu u) = \varepsilon \tag{2.1.30}$$

by means of the change of the dependent variable (2.1.28).

Note 2.1.3. It follows from Theorem 2.1.3 that the nonlinear differential operator $\varepsilon u^2 \Box$ defined on the set of solutions of the PDE $(\partial_\mu u)(\partial^\mu u) = \varepsilon$ has a discrete spectrum, i.e.,

$$\varepsilon u^2 \Box u = N u, \quad N = 0, 1, 2, 3 \tag{2.1.31}$$

and what is more, the spectrum is determined by the dimension of the space of independent variables only. Consequently, the nonlinear additional constraint $(\partial_\mu u)(\partial^\mu u) = \varepsilon$ plays the same role as the boundary conditions in the Sturm-Liouville problem [61].

It is natural to expect that an additional constraint changes the symmetry properties of the d'Alembert equation. This conjecture is confirmed by comparison of results given in the Tables 2.1.1, 2.1.2.

Table 2.1.1. **Symmetry of the nonlinear d'Alembert equation (2.1.1) with $n = 4$**

N	Invariance group	$F(u)$
1.	the Poincaré group $P(1,3)$	arbitrary smooth function
2.	the extended Poincaré group $\widetilde{P}(1,3)$ [123, 137]	$C_1(u + C_2)^k$, $C_1 \exp\{ku\}$
3.	the conformal group $C(1,3)$ [123, 189]	$C_1(u + C_2)^3$

Here $C_1,\ C_2,\ k$ are arbitrary constants.

Table 2.1.2. **Symmetry of the system** $\Box u = F(u),\ \ (\partial_\mu u)(\partial^\mu u) = \lambda$

N	Invariance group	$F(u)$	λ
1.	the Poincaré group $P(1,3)$	arbitrary smooth function	$\lambda \in \mathbb{R}^1$
2.	the extended Poincaré group $\widetilde{P}(1,3)$ [137]	$C_1(u+C_2)^{-1}$	$\lambda \in \mathbb{R}^1$
3.	the conformal group $C(1,3)$ [154, 156]	$3\lambda(u+C_1)^{-1}$	$\lambda \in \mathbb{R}^1$
4.	the generalized Poincaré group $P(1,4)$	0	$\lambda > 0$
5.	the generalized Poincaré group $P(2,3)$	0	$\lambda < 0$
6.	infinite-dimensional group	0	0

Here $C_1,\ C_2$ are arbitrary constants.

Comparing Tables 2.1.1, 2.1.2 we come to the conclusion that the conformally non-invariant nonlinear d'Alembert equation $\Box u = 3u^{-1}$ after being restricted to the set of solutions of the Hamilton equation $(\partial_\mu u)(\partial^\mu u) = 1$ admits the conformal group $C(1,3)$. Consequently, an additional constraint $(\partial_\mu u)(\partial^\mu u) = 1$ "selects" a subset of solutions which is invariant under the group $C(1,3)$. In other words, the nonlinear d'Alembert equation $\Box u = 3u^{-1}$ is *conditionally-invariant* with respect to the conformal group.

Such a definition of conditional invariance is much more general than that introduced in Chapter 1. Indeed, when defining in Section 1.5 a conditional invariance of a given PDE we restricted ourselves to considering additional constraints which were first-order quasi-linear PDEs. It is straightforward to verify that the nonlinear d'Alembert equation mentioned in the previous paragraph is not conditionally-invariant with respect to conformal group in the sense of Definition 1.5.3. Nevertheless, its generalized conditional invariance can be used effectively to construct exact solutions. The peculiarity is that Ansätze invariant under three-dimensional subalgebras of the conformal algebra not belonging to the Lie algebra of the extended Poincaré group reduce the equation $\Box u = 3u^{-1}$ to two ODEs.

But we are not going to apply the symmetry reduction procedure to constructing solutions of the d'Alembert-Hamilton system, since we have devel-

oped a method enabling us to construct its general solution.

2. Integration of the d'Alembert-Hamilton system. It follows from Theorem 2.1.2 that the compatible system of PDEs (2.1.3), (2.1.4) is equivalent either to (2.1.27) or to the following system:

$$\Box u = 0, \quad (\partial_\mu u)(\partial^\mu u) = 0. \tag{2.1.32}$$

General solutions of systems of PDEs (2.1.27), (2.1.32) are given by the following assertions.

Theorem 2.1.4. *The general solution of system of PDEs (2.1.27) is given by one of the following formulae:*
1) $N = 0$,

$$u = A_\mu(\tau)x^\mu + R_1(\tau),$$

where $\tau = \tau(x)$ *is determined in implicit way*

$$B_\mu(\tau)x^\mu + R_2(\tau) = 0$$

and $A_\mu(\tau)$, $B_\mu(\tau)$, $R_1(\tau)$, $R_2(\tau)$ *are arbitrary smooth complex-valued functions satisfying the conditions*

$$A_\mu A^\mu = 1, \quad A_\mu B^\mu = 0, \quad \dot{A}_\mu B^\mu = 0, \quad B_\mu B^\mu = 0;$$

2) $N = 1$,

$$u^2 = (a_\mu x^\mu + G_1)^2 - (b_\mu x^\mu + G_2)^2,$$

where $G_i = G_i(\theta_\mu x^\mu) \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ *are arbitrary functions,* a_μ, b_μ, θ_μ *are arbitrary complex parameters satisfying the conditions*

$$a_\mu a^\mu = -b_\mu b^\mu = 1, \quad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0;$$

3) $N = 2$,

a) $u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big) + \Big\{B_\mu(\tau)\Big(x^\mu + A^\mu(\tau)\Big)\Big\}^2$, *where* $\tau = \tau(x)$ *is determined in implicit way*

$$\Big(x_\mu + A_\mu(\tau)\Big)\dot{B}^\mu(\tau) = 0,$$

$A_\mu(\tau)$, $B_\mu(\tau)$ *are arbitrary smooth complex-valued functions satisfying the conditions*

$$B_\mu B^\mu = -1, \quad \dot{B}_\mu \dot{B}^\mu = 0, \quad \dot{A}_\mu = R(\tau)\dot{B}_\mu$$

with arbitrary $R(\tau) \in C^1(\mathbb{C}^1,\ \mathbb{C}^1)$*;*

b) $u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big) + \Big\{b_\mu\Big(x^\mu + A^\mu(\tau)\Big)\Big\}^2$, *where* $\tau = \tau(x)$ *is determined in implicit way*

$$\Big(x_\mu + A_\mu(\tau)\Big)\dot{A}^\mu(\tau) + \Big(x_\mu + A_\mu(\tau)\Big)b^\mu b_\nu \dot{A}^\nu(\tau) = 0,$$

$A_\mu(\tau)$ *are arbitrary smooth complex-valued functions satisfying the condition*

$$\dot{A}_\mu \dot{A}^\mu + (b_\mu \dot{A}^\mu)^2 = 0,$$

b_μ *are arbitrary complex constants satisfying the condition* $b_\mu b^\mu = -1$*;*

4) $N = 3$,

$$u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big), \tag{2.1.33}$$

where $\tau = \tau(x)$ *is determined in implicit way*

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0, \tag{2.1.34}$$

$A_\mu(\tau),\ B_\mu(\tau)$ *are arbitrary smooth complex-valued functions satisfying the conditions*

$$\dot{A}_\mu B^\mu = 0, \quad B_\mu B^\mu = 0. \tag{2.1.35}$$

Proof. We will give a detailed proof of the theorem for the case $N = 3$. In the remaining cases only the schemes of the proofs will be outlined.

Our approach for integration of the d'Alembert-Hamilton system is based on the generalization of the nonlocal transformation method [145, 146] to a case of multi-dimensional PDEs suggested in [165]–[167].

By a nonlocal transformation of the order r we mean the transformation

$$\begin{aligned} x'_\mu &= f_\mu(x,\, u,\, \underset{1}{u}, \ldots, \underset{k}{u}), \\ u' &= f(x,\, u,\, \underset{1}{u}, \ldots, \underset{k}{u}), \end{aligned} \tag{2.1.36}$$

where $\{f_\mu, f\} \subset C^r(\mathbb{C}^n, \mathbb{C}^1)$, the symbol $\underset{s}{u}$ denotes the set of second-order derivatives of the function $u = u(x)$.

A principal idea of the mentioned method is to linearize a PDE under study by means of the proper nonlocal transformation (2.1.36). If we succeed in constructing a solution of the linear equation (general or particular), then a solution of the initial equation is obtained by inverting transformation (2.1.36).

Especially important are the contact transformations (first-order nonlocal transformations)

$$x'_\mu = f_\mu(x,\, u,\, \underset{1}{u}), \quad u' = f(x,\, u,\, \underset{1}{u}), \quad u'_{x_\mu} = g_\mu(x,\, u,\, \underset{1}{u}), \tag{2.1.37}$$

which preserve the first-order tangency condition

$$du - u_{x_\mu} dx_\mu = 0 \Rightarrow du' - u'_{x'_\mu} dx'_\mu = 0.$$

This fact is explained by that any two first-order PDEs can be transformed one into another by means of a proper contact transformation [190, 218].

According to Lemma 2.1.2 $\det \|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 0$. Consequently, the rank of the matrix $U = \|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 0$ takes the values 1,2,3. Each case listed has to be considered separately.

Case 1. $\operatorname{rank} U = 3$. With such a condition there is a non-vanishing third-order minor of the matrix U. Making, if necessary, changes $x_0 \to ix_a,\ x_a \to ix_0$ or $x_a \to x_b,\ x_b \to x_a$ which leave system (2.1.27) invariant we can choose

$$\det \|u_{x_a x_b}\|^3_{a,b=1} \neq 0. \tag{2.1.38}$$

Performing the generalized Euler-Ampère transformation [165]:

$$\begin{aligned}
&y_0 = x_0, \quad y_a = u_{x_a}, \quad H(y) = x_a u_{x_a} - u,\\
&H_{y_0} = -u_{x_0}, \quad H_{y_a} = x_a, \quad a = 1,2,3,\\
&H_{12} = -\begin{vmatrix} u_{12} & u_{23} \\ u_{13} & u_{33} \end{vmatrix} \Delta^{-1}, \quad H_{11} = \begin{vmatrix} u_{22} & u_{23} \\ u_{23} & u_{33} \end{vmatrix} \Delta^{-1},\\
&H_{31} = -\begin{vmatrix} u_{12} & u_{22} \\ u_{13} & u_{23} \end{vmatrix} \Delta^{-1}, \quad H_{22} = \begin{vmatrix} u_{11} & u_{13} \\ u_{13} & u_{33} \end{vmatrix} \Delta^{-1},\\
&H_{23} = -\begin{vmatrix} u_{11} & u_{12} \\ u_{13} & u_{23} \end{vmatrix} \Delta^{-1}, \quad H_{33} = \begin{vmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{vmatrix} \Delta^{-1},\\
&H_{01} = -\begin{vmatrix} u_{01} & u_{02} & u_{03} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{vmatrix} \Delta^{-1},\\
&H_{02} = -\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{01} & u_{02} & u_{03} \\ u_{13} & u_{23} & u_{33} \end{vmatrix} \Delta^{-1},\\
&H_{03} = -\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{01} & u_{02} & u_{03} \end{vmatrix} \Delta^{-1},\\
&H_{00} = -\Delta^{-1} \det \|u_{\mu\nu}\|^3_{\mu,\nu=0},
\end{aligned} \tag{2.1.39}$$

where $u_{\mu\nu} = u_{x_\mu x_\nu}$, $H_{\mu\nu} = H_{y_\mu y_\nu}$, $|W| = \det\|W\|$, $\Delta = \det\|u_{ab}\|^3_{a,b=1}$, in (2.1.27) we get

$$\begin{aligned}&\det\|H_{y_\mu y_\nu}\|^3_{\mu,\nu=0} + \Sigma_2(H) + 3[T(H)]^{-1}\det\|H_{y_a y_b}\|^3_{a,b=1} = 0,\\ &H_{y_0} = -(1+y_a y_a)^{1/2}.\end{aligned} \tag{2.1.40}$$

Hereafter $T(H) = y_a H_{y_a} - H$, $\Sigma_2(H)$ is the sum of the second-order principal minors of the matrix $\|H_{y_\mu y_\nu}\|^3_{\mu,\nu=0}$.

Thus, instead of the nonlinear Hamilton equation, we have a simple linear PDE which is easily integrated

$$H = -y_0(1+y_a y_a)^{1/2} - B(y_1,y_2,y_3), \tag{2.1.41}$$

where $B \in C^2(\mathbb{C}^3,\mathbb{C}^1)$ is an arbitrary function.

Inserting (2.1.41) into the first equation from (2.1.40) and multiplying by $T(H)$ we note that the equation obtained is rewritten in the following way:

$$a_1 y_0^2 + a_2 y_0 + a_3 = 0, \tag{2.1.42}$$

where

$$\begin{aligned} a_1 &= \triangle_3 B + y_a y_b B_{y_a y_b} + 3T(B),\\ a_2 &= \Sigma_2(B) + y_a y_b B_{y_a y_b}\triangle_3 B - y_a y_b B_{y_a y_c} B_{y_c y_b} - 3[T(B)]^2,\\ a_3 &= (1+y_a y_a)\det\|B_{y_a y_b}\|^3_{a,b=1} + [T(B)]^3.\end{aligned}$$

Since $a_1,\ a_2,\ a_3$ are independent of y_0, from (2.1.42) it follows that $a_1 = a_2 = a_3 = 0$.

Thus, we reduce d'Alembert-Hamilton system (2.1.27) with $\operatorname{rank} U = 3$ to the system of three nonlinear PDEs with three independent variables

$$\begin{aligned} &1)\quad \triangle_3 B + y_a y_b B_{y_a y_b} = -3T(B),\\ &2)\quad \Sigma_2(B) + y_a y_b B_{y_a y_b}\triangle_3 B - y_a y_b B_{y_a y_c} B_{y_c y_b} = 3[T(B)]^2,\\ &3)\quad \det\|B_{y_a y_b}\|^3_{a,b=1} = -[T(B)]^3(1+y_a y_a)^{-1}.\end{aligned} \tag{2.1.43}$$

The above system is simplified substantially by means of the following change of variables:

$$\begin{aligned} &z_a = y_a(1+y_b y_b)^{-1/2},\\ &P(z_1,z_2,z_3) = (1+y_a y_a)^{-1/2} B(y_1,y_2,y_3).\end{aligned} \tag{2.1.44}$$

In the new variables $z,\ P(z)$ system (2.1.43) reads

$$\begin{aligned} &1)\quad \triangle_3 P - z_a z_b P_{z_a z_b} = 0,\\ &2)\quad \Sigma_2(P) - z_a z_b P_{z_a z_b}\triangle_3 P + z_a z_b P_{z_a z_c} P_{z_c z_b} = 0,\\ &3)\quad \det \|P_{z_a z_b}\|^3_{a,b=1} = 0. \end{aligned} \tag{2.1.45}$$

Since $\det \widetilde{P} = \det \|P_{z_a z_b}\|^3_{a,b=1} = 0$, the rank of the (3×3)-matrix $\widetilde{P}$ is equal either to 1 or to 2.

Subcase 1.1. $\operatorname{rank}\widetilde{P} = 1$. Hence, according to the theorem about an implicit function, it follows that there are such functions $\{R_1, R_2\} \subset C^2(\mathbb{C}^1, \mathbb{C}^1)$ that

$$P_{z_k} = R_k(P_{z_3}), \quad k = 1, 2. \tag{2.1.46}$$

Substitution of (2.1.46) into the second equation of system (2.1.45) shows that its left-hand side vanishes under arbitrary $R_1,\ R_2$. The first equation takes the form

$$\Big(1 + \dot{R}_k \dot{R}_k - (z_k \dot{R}_k + z_3)^2\Big) P_{z_3 z_3} = 0,$$

whence

$$P_{z_3 z_3} = 0 \tag{2.1.47}$$

or

$$1 + \dot{R}_k \dot{R}_k - (z_k \dot{R}_k + z_3) = 0, \tag{2.1.48}$$

where $\dot{R}_k = dR_k/dP_{z_3},\ k = 1, 2$. Hereafter in this section, the summation over the repeated indices denoted by the letters $k,\ l,\ n$ from 1 to 2 is understood.

Let the equality (2.1.47) hold true. Then, differentiating (2.1.46) with respect to z_3 we have $P_{z_1 z_3} = P_{z_2 z_3} = 0$. Next, differentiating (2.1.46) with respect to $z_1,\ z_2$ we conclude that $P_{z_a z_b} = 0,\ a, b = 1, 2, 3$, whence

$$P = C_a z_a + C_0,\ C_\mu \in \mathbb{C}^1. \tag{2.1.49}$$

Now we turn to the case $P_{z_3 z_3} \neq 0$. Hence it follows that the equality (2.1.48) holds. To integrate system of the first-order PDEs (2.1.46), (2.1.48) we make the contact transformation

$$\begin{aligned} &t_k = z_k, \quad t_3 = P_{z_3}, \quad G(t_1, t_2, t_3) = z_3 P_{z_3} - P,\\ &G_{t_k} = -P_{z_k}, \quad G_{t_3} = z_3, \quad k = 1, 2. \end{aligned}$$

As a result, we get

$$\begin{aligned} &G_{t_k} = -R_k(t_3), \quad k = 1, 2,\\ &1 + \dot{R}_k(t_3)\dot{R}_k(t_3) - \Big(t_k \dot{R}_k(t_3) + G_{t_3}\Big)^2 = 0. \end{aligned} \tag{2.1.50}$$

Integration of the first two equations of system (2.1.50) yields

$$G = -t_k R_k(t_3) + Q(t_3), \tag{2.1.51}$$

where $Q \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ is an arbitrary function.

Substituting the result obtained into the third equation of system (2.1.50) we have

$$1 + \dot{R}_k \dot{R}_k - \Big(t_k \dot{R}_k - t_k \dot{R}_k + \dot{Q}\Big)^2 \equiv 1 + \dot{R}_k \dot{R}_k - \dot{Q}^2 = 0. \tag{2.1.52}$$

Thus, formulae (2.1.51), (2.1.52) determine the general solution of system of PDEs (2.1.50). Returning to the initial variables z, $P(z)$ we obtain the general solution of system (2.1.46), (2.1.48)

$$P = z_k R_k(t_3) + t_3 z_3 - Q(t_3), \quad 1 + \dot{R}_k \dot{R}_k - \dot{Q}^2 = 0, \tag{2.1.53}$$

where $t_3 = t_3(z)$ is determined by the relation $G_{t_3} = z_3$, whence

$$z_k \dot{R}_k(t_3) + z_3 - \dot{Q}(t_3) = 0. \tag{2.1.54}$$

To represent formulae (2.1.53), (2.1.54) in a manifestly $O(3)$-invariant form we re-determine the parametric function to be $t_3(z) = \widetilde{R}_3\Big(\tau(z)\Big)$ and designate

$$\widetilde{R}_k(\tau) = R_k\Big(\widetilde{R}_3(\tau)\Big), \quad \widetilde{Q}(\tau) = -Q\Big(\widetilde{R}_3(\tau)\Big), \quad k = 1, 2.$$

With such notations formulae (2.1.53), (2.1.54) read

$$P = z_a \widetilde{R}_a(\tau) + \widetilde{Q}(\tau), \quad \dot{\widetilde{R}}_a \dot{\widetilde{R}}_a - \dot{\widetilde{Q}}^2 = 0, \tag{2.1.55}$$

where $\tau = \tau(z)$ is a smooth function defined by the relation

$$z_a \dot{\widetilde{R}}_a(\tau) + \dot{\widetilde{Q}}(\tau) = 0. \tag{2.1.56}$$

Thus, the general solution of system of PDEs (2.1.45) is given by one of formulae (2.1.49) or (2.1.55), (2.1.56). Making the change of variables (2.1.44) we obtain the general solution of the system of nonlinear PDEs (2.1.43)

$$\begin{aligned} B(y) &= C_a y_a + C_0 (1 + y_a y_a)^{1/2}, && (2.1.57) \\ B(y) &= y_a \widetilde{R}_a(\tau) + \widetilde{Q}(\tau)(1 + y_a y_a)^{1/2}, \quad \dot{\widetilde{R}}_a \dot{\widetilde{R}}_a - \dot{\widetilde{Q}}^2 = 0, && (2.1.58) \end{aligned}$$

where $\tau = \tau(y)$ is a smooth function determined in implicit way

$$y_a \dot{\widetilde{R}}_a(\tau) + \dot{\widetilde{Q}}(\tau)(1 + y_a y_a)^{1/2} = 0. \tag{2.1.59}$$

Evidently, the solution (2.1.57) is contained in the class (2.1.58), (2.1.59). Inserting the expression for the function $B(y)$ from (2.1.58) into (2.1.41) we have

$$H(y) = -(1 + y_a y_a)^{1/2} \Big(y_0 + \widetilde{Q}(\tau) \Big) - y_a \widetilde{R}_a(\tau),$$

where the function $\tau = \tau(y)$ is determined by (2.1.59).

At last, rewriting the expression obtained in the initial variables x, $u(x)$ we arrive at the following class of solutions of the d'Alembert-Hamilton system (2.1.27):

$$u(x) = x_a y_a - H = \Big(x_a + \widetilde{R}_a(\tau) \Big) y_a + (1 + y_a y_a)^{1/2} \Big(x_0 + \widetilde{Q}(\tau) \Big), \tag{2.1.60}$$

where $y_a = y_a(x)$ are determined by the equalities

$$x_a = H_{y_a} = -\widetilde{R}_a(\tau) - y_a(1 + y_b y_b)^{-1/2} \Big(x_0 + \widetilde{Q}(\tau) \Big), \quad a = 1, 2, 3.$$

Resolving the above equalities with respect to y_a we get

$$y_a = -(x_a + \widetilde{R}_a) \Big((x_0 + \widetilde{Q})^2 - (x_b + \widetilde{R}_b)(x_b + \widetilde{R}_b) \Big)^{-1/2}.$$

Substitution of the expressions obtained into (2.1.60) yields

$$u(x) = \Big((x_0 + \widetilde{Q})^2 - (x_b + \widetilde{R}_b)(x_b + \widetilde{R}_b) \Big)^{1/2},$$

where $\tau = \tau(x)$ is a smooth function determined by the equality

$$y_a \dot{\widetilde{R}}_a(\tau) + \dot{\widetilde{Q}}(\tau)(1 + y_a y_a)^{1/2} \equiv \Big(x_0 + \widetilde{Q}(\tau) \Big) \dot{\widetilde{Q}}(\tau) - \Big(x_a + \widetilde{R}_a(\tau) \Big) \dot{\widetilde{R}}_a(\tau) = 0$$

and $\widetilde{Q}$, $\widetilde{R}_a$ are arbitrary smooth functions satisfying the relation $\dot{\widetilde{R}}_a \dot{\widetilde{R}}_a - \dot{\widetilde{Q}}^2 = 0$. Introducing the notations $A_0 = \widetilde{Q}$, $A_a = \widetilde{R}_a$ we obtain formulae (2.1.33)–(2.1.35) under $B_\mu \equiv \dot{A}_\mu$, $\mu = 0, \ldots, 3$.

Subcase 1.2. $\operatorname{rank} \widetilde{P} = 2$. Without loss of generality, we can assume that

$$\det \begin{Vmatrix} P_{z_1 z_1} & P_{z_1 z_2} \\ P_{z_1 z_2} & P_{z_2 z_2} \end{Vmatrix} \neq 0.$$

Consequently, there is such a function $R \in C^3(\mathbb{C}^2, \mathbb{C}^1)$ that the relation $P_{z_3} = R(P_{z_1},\, P_{z_2})$ holds. With account of this fact system (2.1.45) is rewritten in the following way:

$$
\begin{aligned}
&P_{z_k z_k} + (z_k + z_3 R_k)(z_n + z_3 R_n) P_{z_k z_n} = 0,\\
&(1 - z_k z_k - z_3^2)(1 + R_k R_k) + (z_3 - z_k R_k)^2 = 0, \qquad (2.1.61)\\
&P_{z_3} = R(P_{z_1},\, P_{z_2}).
\end{aligned}
$$

Here $R_k = \partial R/\partial(P_{z_k}),\ k = 1, 2.$

Let us perform in (2.1.61) the following contact transformation:

$$
\begin{aligned}
&t_k = P_{z_k}, \quad t_3 = z_3, \quad G(t_1, t_2, t_3) = z_k P_{z_k} - P,\\
&G_{t_k} = z_k, \quad G_{t_3} = -P_{z_3}, \quad k = 1, 2,\\
&G_{t_1 t_1} = \delta^{-1} P_{z_2 z_2}, \quad G_{t_1 t_2} = -\delta^{-1} P_{z_1 z_2},\\
&G_{t_2 t_2} = \delta^{-1} P_{z_1 z_1}, \quad G_{t_3 t_3} = -\delta^{-1} \det \|P_{z_a z_b}\|_{a,b=1}^{3},\\
&G_{t_1 t_3} = \delta^{-1}(P_{z_1 z_2} P_{z_2 z_3} - P_{z_2 z_2} P_{z_1 z_3}),\\
&G_{t_2 t_3} = \delta^{-1}(P_{z_1 z_2} P_{z_1 z_3} - P_{z_1 z_1} P_{z_2 z_3}),
\end{aligned}
$$

where $\delta = P_{z_1 z_1} P_{z_2 z_2} - P_{z_1 z_2}^2 \neq 0.$

Being rewritten in the new variables $t,\ G(t)$ system (2.1.61) takes the form

$$
\begin{aligned}
1)\quad &\Big(1 + R_{t_2}^2 - (G_{t_2} + t_3 R_{t_2})^2\Big) G_{t_1 t_1} - 2\Big(R_{t_1} R_{t_2} - (G_{t_1} + t_3 R_{t_1})\\
&\times (G_{t_2} + t_3 R_{t_2})\Big) G_{t_1 t_2} + \Big(1 + R_{t_1}^2 - (G_{t_1} + t_3 R_{t_1})^2\Big) G_{t_2 t_2} = 0,\\
2)\quad &(1 - t_3^2 - G_{t_k} G_{t_k})(1 + R_{t_k} R_{t_k}) + (t_3 - R_{t_k} G_{t_k})^2 = 0, \qquad (2.1.62)\\
3)\quad &G_{t_3} = R(t_1, t_2).
\end{aligned}
$$

Integrating equation 3 from (2.1.62) we have

$$
G = -t_3 R(t_1, t_2) + iQ(t_1, t_2), \qquad (2.1.63)
$$

where $Q \in C^3(\mathbb{C}^2, \mathbb{C}^1)$ is an arbitrary function.

Substituting the expression (2.1.63) into the equations 1,2 from (2.1.62) and splitting with respect to the variable t_3 we arrive at the two-dimensional system of PDEs for the functions $R(t_1, t_2),\ Q(t_1, t_2)$:

$$
\begin{aligned}
1)\quad &(1 + Q_{t_k} Q_{t_k})(1 + R_{t_n} R_{t_n}) - (Q_{t_k} R_{t_k})^2 = 0, \qquad (2.1.64)\\
2)\quad &(1 + Q_{t_k} Q_{t_k} + R_{t_k} R_{t_k}) \triangle_2 Q - (Q_{t_k} Q_{t_n} + R_{t_k} R_{t_n}) Q_{t_k t_n} = 0,\\
3)\quad &(1 + Q_{t_k} Q_{t_k} + R_{t_k} R_{t_k}) \triangle_2 R - (Q_{t_k} Q_{t_n} + R_{t_k} R_{t_n}) R_{t_k t_n} = 0,
\end{aligned}
$$

where $\triangle_2 = \partial_{t_1}^2 + \partial_{t_2}^2$.

We have succeeded in integrating the over-determined system (2.1.64). Making use of formulae (2.1.41), (2.1.44), (2.1.63), we rewrite its general solution in the initial variables $x,\ u(x)$. After representing the result obtained in a manifestly covariant form we arrive at formulae (2.1.33)–(2.1.35).

Case 2. $\operatorname{rank} U < 3$. When studying the compatibility of the d'Alembert-Hamilton system, we have established that system of PDEs (2.1.27) with $N = 3$ is incompatible provided $\operatorname{rank} \|u_{x_\mu x_\nu}\|_{\mu,\nu=0}^3 < 3$ [165].

Consequently, any solution of system (2.1.27) can be reduced by means of one of the transformations $x_0 \to ix_a,\ x_a \to ix_0$ or $x_a \to x_b,\ x_b \to x_a$ to the form (2.1.33)–(2.1.35). Since the class of functions $u(x)$ determined by formulae (2.1.33)–(2.1.35) is invariant with respect to the above transformations, hence it follows that any solution of the d'Alembert-Hamilton system (2.1.27) with $N = 3$ is contained in it. To complete the proof for the case $N = 3$ it suffices to check that any function $u(x)$ determined by (2.1.33)–(2.1.35) satisfies the d'Alembert-Hamilton system (2.1.27). The check is performed by direct computation. Differentiating the equalities (2.1.33), (2.1.34) with respect to x_μ and excluding from the equalities obtained τ_{x_μ} we get

$$u_{x_\mu} = \Big((x_\nu + A_\nu)(x^\nu + A^\nu)\Big)^{-1/2}\Big(x^\mu + A^\mu - \rho(\dot{A}\cdot x + \dot{A}\cdot A)B^\mu\Big), \quad (2.1.65)$$

where $A \cdot x = A_\mu x^\mu$, $\rho = (\dot{B}\cdot x + \dot{B}\cdot A)^{-1}$.

Since

$$\begin{aligned} g_{\mu\nu} u_{x_\mu} u_{x_\nu} &= \Big((x_\nu + A_\nu)(x^\nu + A^\nu)\Big)^{-1}\Big(x_\mu + A_\mu - \rho(\dot{A}\cdot x + \dot{A}\cdot A)B_\mu\Big) \\ &\quad \times\Big(x^\mu + A^\mu - \rho(\dot{A}\cdot x + \dot{A}\cdot A)B^\mu\Big) = 1 \end{aligned}$$

(we have used the equalities (2.1.35)), the Hamilton equation is identically satisfied.

Next, differentiating (2.1.65) with respect to x_μ and excluding τ_{x_μ} we get

$$\begin{aligned} u_{x_\mu x_\nu} &= -\Big((x_\alpha + A_\alpha)(x^\alpha + A^\alpha)\Big)^{-3/2}\Big(x^\mu + A^\mu - \rho(\dot{A}\cdot x + \dot{A}\cdot A)B^\mu\Big) \\ &\quad \times\Big(x^\nu + A^\nu - \rho(\dot{A}\cdot x + \dot{A}\cdot A)B^\nu\Big)\Big((x_\alpha + A_\alpha)(x^\alpha + A^\alpha)\Big)^{-1/2} \\ &\quad \times\Big(g_{\mu\nu} - \rho(\dot{A}^\mu B^\nu + \dot{A}^\nu B^\mu) + \rho^2[\ddot{A}\cdot x + \ddot{A}\cdot A + \dot{A}\cdot\dot{A} \\ &\quad +(\dot{A}\cdot x + \dot{A}\cdot A)(\ddot{B}\cdot x + \ddot{B}\cdot A + \dot{B}\cdot\dot{A})]B^\mu B^\nu \\ &\quad +\rho^2(\dot{A}\cdot x + \dot{A}\cdot A)(\dot{B}^\mu B^\nu + \dot{B}^\nu B^\mu)\Big). \end{aligned}$$

Convoluting $u_{x_\mu x_\nu}$ with the metric tensor $g_{\mu\nu}$ and taking into account the equalities (2.1.34) we come to the following relation:

$$\Box u = \Big((x_\alpha + A_\alpha)(x^\alpha + A^\alpha)\Big)^{-1/2}(g_{\mu\nu}g_{\mu\nu} - 1) = 3u^{-1},$$

the same as what was to be proved.

Further, we will outline the scheme of the proof of the theorem provided $N = 0, 1, 2$ in (2.1.27).

According to [165] system of PDEs (2.1.27) with $N = 2$ is compatible if and only if rank $\|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 2$. Consequently, without loss of generality, we can suppose that the condition

$$\delta = \begin{vmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2} \end{vmatrix} \neq 0$$

holds.

Since rank $\|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 2$ and $\delta \neq 0$, there exists such a function $S \in C^2(\mathbb{C}^4, \mathbb{C}^1)$ that solutions of the d'Alembert-Hamilton equation (2.1.27) with $N = 2$ satisfy an additional constraint $S(u_{x_0}, u_{x_1}, u_{x_2}, u_{x_3}) = 0$. Consequently, in the case involved we have to solve the following over-determined system of PDEs:

$$\Box u = 2u^{-1}, \quad (\partial_\mu u)(\partial^\mu u) = 1, \quad S(u_{x_0}, u_{x_1}, u_{x_2}, u_{x_3}) = 0.$$

Due to the condition $\delta \neq 0$ we can resolve the last two equations with respect to u_{x_0}, u_{x_3} and rewrite the above system as follows

$$\begin{aligned} u_{x_0} &= \Big(1 + u_{x_k}u_{x_k} + W^2(u_{x_1}, u_{x_2})\Big)^{1/2}, \\ u_{x_3} &= W(u_{x_1}, u_{x_2}), \quad \Box u = 2u^{-1}. \end{aligned} \tag{2.1.66}$$

Let us apply to the system of PDEs (2.1.66) the contact transformation

$$\begin{aligned} & x_0 = y_0, \quad x_k = H_{y_k}, \quad x_3 = y_3, \\ & u = y_k H_{y_k} - H, \\ & u_{x_0} = -H_{y_0}, \quad u_{x_k} = y_k, \quad u_{x_3} = -H_{y_3}, \\ & H_{11} = u_{22}\delta^{-1}, \quad H_{12} = -u_{12}\delta^{-1}, \quad H_{22} = u_{11}\delta^{-1}, \\ & H_{01} = -\begin{vmatrix} u_{01} & u_{12} \\ u_{02} & u_{22} \end{vmatrix}\delta^{-1}, \quad H_{23} = -\begin{vmatrix} u_{11} & u_{13} \\ u_{12} & u_{23} \end{vmatrix}\delta^{-1}, \\ & H_{13} = -\begin{vmatrix} u_{13} & u_{12} \\ u_{23} & u_{22} \end{vmatrix}\delta^{-1}, \quad H_{02} = -\begin{vmatrix} u_{11} & u_{01} \\ u_{12} & u_{02} \end{vmatrix}\delta^{-1}, \end{aligned} \tag{2.1.67}$$

$$H_{00} = -\begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{01} & u_{11} & u_{12} \\ u_{02} & u_{12} & u_{22} \end{vmatrix} \delta^{-1}, \quad H_{03} = -\begin{vmatrix} u_{01} & u_{02} & u_{03} \\ u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \end{vmatrix} \delta^{-1},$$
$$H_{33} = -\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{vmatrix} \delta^{-1}.$$

Here $H_{\mu\nu} = \partial^2 H/\partial y_\mu \partial y_\nu,\ u_{\mu\nu} = \partial^2 u/\partial x_\mu \partial x_\nu,\ \mu, \nu = 0, \ldots, 3.$

The first two equations of system (2.1.67) are linearized by the transformation (2.1.67)

$$\begin{aligned} H_{y_0} &= -\Big(1 + y_k y_k + W^2(y_1, y_2)\Big)^{1/2}, \\ H_{y_3} &= -W(y_1, y_2). \end{aligned}$$

Integrating the above system, inserting the obtained expression for $H(y)$

$$H = -y_0\Big(1 + y_k y_k + W^2(y_1, y_2)\Big)^{1/2} - y_3 W(y_1, y_2) - B(y_1, y_2), \qquad (2.1.68)$$

where $B \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ is an arbitrary function, into the last equation of system (2.1.66) and splitting with respect to $y_0,\ y_3$ we come to the over-determined system of five PDEs for two functions $B(y_1, y_2),\ W(y_1, y_2)$

$$\begin{aligned}
1)\quad & (\triangle_2 W + y_k y_n W_{y_k y_n})\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big) \\
& -\Big(T(W) y_k + W_{y_k}\Big)\Big(T(W) y_n + W_{y_n}\Big) W_{y_k y_n} \\
& = -2T(W)\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big), \\
2)\quad & \det \|W_{y_k y_n}\|^2_{k,n=1} = T^2(W)\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big)^{-1} \\
& \times (1 + y_k y_k + W^2)^{-1}, \\
3)\quad & (\triangle_2 B + y_k y_n B_{y_k y_n})\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big) \\
& -\Big(T(W) y_k + W_{y_k}\Big)\Big(T(W) y_n + W_{y_n}\Big) B_{y_k y_n} \\
& = -2T(B)\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big), \\
4)\quad & \det \|B_{y_k y_n}\|^2_{k,n=1} = T^2(B)\Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big) \\
& \times (1 + y_k y_k + W^2)^{-1}, \\
5)\quad & (\triangle_2 W)(\triangle_2 B) - B_{y_k y_n} W_{y_k y_n} = T(W) T(B) \\
& \times \Big(1 + W_{y_k} W_{y_k} + T^2(W)\Big)(1 + y_k y_k + W^2)^{-1}.
\end{aligned}$$

Here the notations $T(F) = y_k F_{y_k} - F,\ \triangle_2 F = F_{y_k y_k}$ are used.

Integrating the above system and returning to the initial variables $x,\ u(x)$ according to the formulae (2.1.67) we get the general solution of the d'Alembert-Hamilton system (2.1.27) with $N = 2$ which is contained in the class of functions $u(x)$ determined by the formulae 3 from the statement of Theorem 2.1.4.

According to [165] system of PDEs (2.1.27) with $N = 1$ is compatible only in the following cases

$a)$ rank $\|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 2$;

$b)$ rank $\|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 1$.

In the case a, we apply to the system under study the contact transformation (2.1.67). The general solution of the Hamilton equation being written in the variables $y,\ H(y)$ takes the form (2.1.68) with arbitrary smooth functions $B(y_1, y_2),\ W(y_1, y_2)$. Inserting (2.1.68) into the d'Alembert equation written in the variables $y,\ H(y)$ and splitting the equality obtained with respect to $y_0,\ y_3$ we arrive at the following system of four PDEs:

$$
\begin{aligned}
&1) \quad 1 + W_{y_k} W_{y_k} + T^2(W) = 0,\\
&2) \quad \det \|W_{y_k y_n}\|^2_{k,n=1} = 0,\\
&3) \quad (1 + y_k y_k + W^2) \det \|B_{y_k y_n}\|^2_{k,n=1} = T(B)\\
&\qquad \times \Big(T(W) y_k + W_{y_k}\Big)\Big(T(W) y_n + W_{y_n}\Big) B_{y_k y_n},\\
&4) \quad (1 + y_k y_k + W^2)\Big((\triangle_2 W)(\triangle_2 B) - B_{y_k y_n} W_{y_k y_n}\Big)\\
&\qquad = T(W)\Big(T(W) y_k + W_{y_k}\Big)\Big(T(W) y_n + W_{y_n}\Big) B_{y_k y_n}.
\end{aligned}
$$

Integrating these equations and returning to the initial variables $x,\ u(x)$ yield the general solution of system (2.1.27) with $N = 1$ provided the condition a holds.

Let us turn now to the case b. Since rank $\|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 1$, there exist such functions $W_a = W_a(u_{x_0}) \in C^1(\mathbb{C}^1, \mathbb{C}^1)$ that

$$u_{x_a} = W_a(u_{x_0}), \quad a = 1, 2, 3.$$

With this remark system (2.1.27) is rewritten in the form

$$\Box u = u^{-1}, \quad u_{x_a} = W_a(u_{x_0}), \tag{2.1.69}$$

where $W_a(\tau)$ are arbitrary smooth functions satisfying the equality $\tau^2 - W_a(\tau) \times W_a(\tau) = 1$.

Make in (2.1.69) the following contact transformation:

$$\begin{aligned}
&y_0 = u_{x_0}, \quad y_a = x_a, \quad H = x_0 u_{x_0} - u,\\
&H_{y_0} = x_0, \quad H_{y_a} = -u_{x_a},\\
&H_{00} = u_{00}^{-1}, \quad H_{0a} = -u_{0a}u_{00}^{-1},\\
&H_{ab} = (u_{0a}u_{0b} - u_{00}u_{ab})u_{00}^{-1}.
\end{aligned} \tag{2.1.70}$$

Here $u_{\mu\nu} = u_{x_\mu x_\nu},\ H_{\mu\nu} = H_{y_\mu y_\nu},\ a, b = 1, 2, 3.$
The last three equations from (2.1.69) are linearized

$$H_{y_a} = -W_a(y_0), \quad a = 1, 2, 3.$$

Inserting the general solution of the above system

$$H = y_a W_a(y_0) - B(y_0), \tag{2.1.71}$$

where $B \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ is an arbitrary function, into the first equation of the system of PDEs (2.1.69) and splitting with respect to y_0 we come to the system of ODEs for the functions $W_a,\ B$

$$\begin{aligned}
&1)\quad \ddot{W}_a = (1 - \dot{W}_b\dot{W}_b)(y_0\dot{W}_a - W_a), \quad a = 1, 2, 3,\\
&2)\quad \ddot{B} = (1 - \dot{W}_b\dot{W}_b)(y_0\dot{B} - B)
\end{aligned}$$

and what is more $W_a W_a = y_0^2 - 1$.

Integrating the system of ODEs obtained and returning to the initial variables $x,\ u(x)$ we obtain a particular case of the formulae 2 from the statement of Theorem 2.1.4.

Provided $N = 0$, the general solution of system of PDEs (2.1.27) splits into two classes satisfying one of the conditions: rank $\|u_{x_\mu x_\nu}\|_{\mu,\nu=0}^3 = 1, 2$.

If rank $\|u_{x_\mu x_\nu}\|_{\mu,\nu=0}^3 = 2$, then we can apply the contact transformation (2.1.67). The general solution of the d'Alembert-Hamilton system is given by the formula (2.1.68), where $B(y_1, y_2),\ W(y_1, y_2)$ are solutions of the system of two PDEs

$$\begin{aligned}
&1)\quad 1 + W_{y_k}W_{y_k} + (y_k W_{y_k} - W)^2 = 0,\\
&2)\quad \Big(y_k(y_n W_{y_n} - W) + W_{y_k}\Big)\Big(y_l(y_n W_{y_n} - W) + W_{y_l}\Big)B_{y_k y_l} = 0.
\end{aligned}$$

Integrating it and returning to the initial variables $x,\ u(x)$ we arrive at the formulae 1 from the statement of Theorem 2.1.4.

Provided solutions of the d'Alembert-Hamilton system (2.1.27) with $N = 0$ satisfy the condition $\operatorname{rank} \|u_{x_\mu x_\nu}\|^3_{\mu,\nu=0} = 1$, we can perform the contact transformation (2.1.70). The general solution of the system obtained is of the form (2.1.71), where $W_a(y_0),\ B(y_0)$ are solutions of the system of two ODEs $\dot{W}_a\dot{W}_a = 1,\ W_aW_a = y_0^2 - 1$. Rewriting (2.1.71) in the initial variables $x,\ u(x)$ according to the formulae (2.1.70) yields the formulae 1 from the statement of Theorem 2.1.4 with $B_\mu \equiv \dot{A}_\mu,\ R_2 = \dot{R}_1$.

Thus, we have established that the general solutions of the system of PDEs (2.1.27) with $N = 0, 1, 2$ are contained in the classes of functions given by the formulae 1–3 from the statement of Theorem 2.1.4. To complete the proof we have to check that the function $u(x)$ determined by these formulae satisfies the d'Alembert-Hamilton system. This check is carried out by direct computation. The theorem is proved. $\triangleright$

Theorem 2.1.5. *The general solution of system of PDEs (2.1.32) has the form*

$$A_\mu(u,\,\tau)x^\mu + A(u,\,\tau) = 0, \tag{2.1.72}$$

where $\tau = \tau(x, u)$ is determined in implicit way

$$B_\mu(u,\tau)x^\mu + B(u,\,\tau) = 0 \tag{2.1.73}$$

and $A_\mu(u,\,\tau),\ B_\mu(u,\,\tau),\ A(u,\,\tau),\ B(u,\,\tau)$ are arbitrary complex-valued functions satisfying the conditions

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0, \quad B_\mu \frac{\partial A^\mu}{\partial \tau} = 0. \tag{2.1.74}$$

Proof. If $u(x) \neq \text{const}$, then making, when necessary, the change of independent variables

$$\begin{gathered} x_0 \to ix_a,\ x_a \to ix_0, \\ x_b \to x_c, \quad x_c \to x_b \end{gathered} \tag{2.1.75}$$

with some fixed $a, b, c = 1, 2, 3$ we can without loss of generality suppose that $u_{x_3} \neq 0$. With this condition we can make in (2.1.32) the hodograph transformation

$$y_\alpha = x_\alpha, \quad \alpha = 0, 1, 2, \quad y_3 = u, \quad U = x_3, \tag{2.1.76}$$

where $y_0, \ldots, y_3$ are new independent variables and $U = U(y)$ is a new dependent variable. As a result, the following system of PDEs

$$U_{y_0y_0} - U_{y_1y_1} - U_{y_2y_2} = 0, \quad U^2_{y_0} - U^2_{y_1} - U^2_{y_2} = 1 \tag{2.1.77}$$

is obtained.

Therefore, the four-dimensional system of PDEs (2.1.32) is transformed to the system with three independent variables (the fourth variable y_3 is contained in (2.1.77) as a parameter).

Equations (2.1.77) are obtained from the d'Alembert-Hamilton system (2.1.27) with $N = 0$ by assuming that its solutions do not depend on x_3 and by identifying x_α with y_α, $\alpha = 0, 1, 2$ and u with U. Consequently, the general solution of (2.1.77) is given by the formulae 1 from the statement of Theorem 2.1.4 provided the indices take the values $0, 1, 2$. And what is more, all arbitrary functions included into the general solution contain y_3 as an argument.

Thus, the general solution of system of PDEs (2.1.77) is determined by the formulae

$$
\begin{aligned}
&U = a_0(\tau,\, y_3)y_0 - a_1(\tau,\, y_3)y_1 - a_2(\tau,\, y_3)y_2 + R_1(\tau,\, y_3),\\
&b_0(\tau,\, y_3)y_0 - b_1(\tau,\, y_3)y_1 - b_2(\tau,\, y_3)y_2 + R_2(\tau,\, y_3) = 0,
\end{aligned}
$$

where $a_\alpha(\tau,\, y_3)$, $b_\alpha(\tau,\, y_3)$, $\alpha = 0, 1, 2$ are arbitrary complex-valued functions satisfying the equalities

$$
\begin{aligned}
&a_0^2 - a_1^2 - a_2^2 = 1, \quad b_0^2 - b_1^2 - b_2^2 = 0,\\
&a_0 b_0 - a_1 b_1 - a_2 b_2 = 0, \quad \frac{\partial a_0}{\partial \tau} b_0 - \frac{\partial a_1}{\partial \tau} b_1 - \frac{\partial a_2}{\partial \tau} b_2 = 0.
\end{aligned}
\tag{2.1.78}
$$

Rewriting the result obtained in the initial variables x, $u(x)$ according to (2.1.76) we arrive at the following representation of the general solution of system (2.1.32):

$$
x_3 = a_0(\tau,\, u)x_0 - a_1(\tau,\, u)x_1 - a_2(\tau,\, u)x_2 + R_1(\tau,\, u),
$$

where $\tau = \tau(x, u)$ is a complex-valued function defined implicitly

$$
b_0(\tau,\, u)x_0 - b_1(\tau,\, u)x_1 - b_2(\tau,\, u)x_2 + R_2(\tau,\, u) = 0
$$

and $a_\alpha(\tau,\, u)$, $b_\alpha(\tau,\, u)$, $\alpha = 0, 1, 2$ are arbitrary complex-valued functions satisfying (2.1.78).

It is readily seen that the above formulae are obtained from (2.1.72)–(2.1.74) under

$$
\begin{aligned}
&A_\alpha = a_\alpha, \quad A_3 = 1, \quad A = R_1,\\
&B_\alpha = b_\alpha, \quad B_3 = 0, \quad B = R_2,
\end{aligned}
$$

where $\alpha = 0, 1, 2$.

We have proved that any solution of system (2.1.32) satisfying the relation $u(x) \neq \text{const}$ can be reduced to the form (2.1.72)–(2.1.74) by the change of the independent variables (2.1.75). Since the class of functions F determined by the relations (2.1.72)–(2.1.74) is invariant with respect to the transformations (2.1.75) and contains the solution $u(x) = \text{const}$, hence it follows that $G \subset F$, where G is the class of functions $u(x)$ determining the general solution of system of PDEs (2.1.32). Let us prove the inverse inclusion $G \subset F$. This assertion will be established if we show that any function $u(x)$ determined by the formulae (2.1.72)–(2.1.74) satisfies equations (2.1.32).

Differentiating equalities (2.1.72), (2.1.73) with respect to x_μ we find u_{x_μ} and τ_{x_μ} as

$$\begin{aligned} u_{x_\mu} &= \frac{1}{\Delta}\Big((x \cdot B_\tau + R_{2\tau})A^\mu - (x \cdot A_\tau + R_{1\tau})B^\mu\Big), \\ \tau_{x_\mu} &= \frac{1}{\Delta}\Big((x \cdot A_u + R_{1u})B^\mu - (x \cdot B_u + R_{2u})A^\mu\Big), \end{aligned} \tag{2.1.79}$$

where $\Delta = (x \cdot A_\tau + R_{1\tau})(x \cdot B_u + R_{2u}) - (x \cdot A_u + R_{1u})(x \cdot B_\tau + R_{2\tau}),\ x \cdot A = x_\mu A^\mu$.

Since

$$\begin{aligned} u_{x_\mu} u_{x^\mu} &= \Delta^{-2}\Big((x \cdot B_\tau + R_{2\tau})^2 A \cdot A - 2(x \cdot B_\tau + R_{2\tau}) \\ &\quad \times (x \cdot A_\tau + R_{1\tau}) A \cdot B + (x \cdot A_\tau + R_{1\tau})^2 B \cdot B\Big) = 0 \end{aligned}$$

(we have used the identities (2.1.74)), the Hamilton equation is satisfied.

Differentiating the first equation from (2.1.79) with respect to x_ν we get

$$\begin{aligned} u_{x_\mu x_\nu} &= -\frac{1}{\Delta^2}\Big((x \cdot B_\tau + R_{2\tau})A^\mu - (x \cdot A_\tau + R_{1\tau})B^\mu\Big) \\ &\quad \times \left(\frac{\partial \Delta}{\partial \tau}\tau_{x_\nu} + \frac{\partial \Delta}{\partial u} u_{x_\nu}\right) + \frac{1}{\Delta}(A^\mu B^\nu_\tau - A^\nu B^\mu_\tau) \\ &\quad + \frac{1}{\Delta}\Big\{\tau_{x_\nu}\frac{\partial}{\partial \tau}\Big((x \cdot B_\tau + R_{2\tau})A^\mu - (x \cdot A_\tau + R_{1\tau})B^\mu\Big) \\ &\quad + u_{x_\nu}\frac{\partial}{\partial u}\Big((x \cdot B_\tau + R_{2\tau})A^\mu - (x \cdot A_\tau + R_{1\tau})B^\mu\Big)\Big\}. \end{aligned}$$

Convoluting $u_{x_\mu x_\nu}$ with the metric tensor $g_{\mu\nu}$ and taking into account identities (2.1.74) we arrive at the equality $\Box u = 0$.

Thus, we have established that the relations $F \subset G,\ G \subset F$ hold, whence it follows that $F = G$. In other words, formulae (2.1.72)–(2.1.74) (the class F)

give the general solution of the d'Alembert-Hamilton equation (2.1.32) (the class G). The theorem is proved. $\rhd$

Note 2.1.4. Assuming that the functions $A_\mu,\ B_\mu$ do not depend on τ and excluding τ from the relations (2.1.72), (2.1.73) we get the following class of the exact solutions of system (2.1.32):

$$g\Big(A_\mu(u)x^\mu,\, B_\mu(u)x^\mu,\, u\Big) = 0, \tag{2.1.80}$$

where $g \in C^2(\mathbb{C}^3, \mathbb{C}^1)$ is an arbitrary function.

Provided $A_\mu,\ B_\mu$ are constants, formula (2.1.80) gives the class of exact solutions of the d'Alembert-Hamilton system obtained by Erugin [77].

Furthermore, if the function g does not depend on $B_\mu(u)x^\mu$, we can resolve (2.1.32) with respect to $A_\mu(u)x^\mu$ and thus get the generalization of the Jacobi-Smirnov-Sobolev formula (2.1.5)

$$A_\mu(u)x^\mu + A(u) = 0, \quad A_\mu A^\mu = 0. \tag{2.1.81}$$

It has been proved in [168, 317] that formulae (2.1.81), where indices take the values $0, 1, \ldots, n-1$, give the general solution of the d'Alembert-Hamilton system $\Box_n u = 0$, $(\partial_A u)(\partial^A u) = 0$, provided u is a real-valued function of n real variables $x_0,\, x_1,\, \ldots,\, x_{n-1}$.

Note 2.1.5. If we choose in (2.1.72)–(2.1.74)

$$A_\mu = C_\mu(\tau), \quad B_\mu = \dot{C}_\mu(\tau), \quad A = C(\tau), \quad B = \dot{C}(\tau),$$

then we get the class of exact solutions

$$u = C_\mu(\tau)x^\mu + C(\tau), \quad \dot{C}_\mu(\tau)x^\mu + \dot{C}(\tau) = 0, \quad \dot{C}_\mu \dot{C}^\mu = 0$$

which was constructed by Bateman [27].

3. Explicit solutions of the d'Alembert-Hamilton system. Theorems 2.1.4, 2.1.5 give a description of the general solution of systems of nonlinear PDEs (2.1.27), (2.1.32) in the parametric form. But for some special choices of the arbitrary functions it is possible to obtain particular solutions in explicit form which is very important for applications of the above results. Below we will construct some real solutions of system (2.1.30) using Theorem 2.1.4.

Take, for example, system (2.1.30) with $N = 3,\ \varepsilon = -1$

$$\Box u = -3u^{-1}, \quad (\partial_\mu u)(\partial^\mu u) = -1. \tag{2.1.82}$$

To obtain the general solution of (2.1.82) it is necessary to make in (2.1.33), (2.1.34), (2.1.35) the change $u \to iu$. As a result, we get the following formulae:

$$\begin{aligned} & u^2 = -\Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big), \\ & \Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0, \\ & B_\mu \dot{A}^\mu = 0, \quad B_\mu B^\mu = 0. \end{aligned} \tag{2.1.83}$$

Putting in the above formulae $A_\mu = 0$, $B_\mu = 0$ we get the well-known $O(1,3)$-invariant solution of system (2.1.27) with $N = 3$: $u(x) = (x_\mu x^\mu)^{1/2}$. This solution can be obtained by means of the symmetry reduction of PDE (2.1.27) with the use of the $O(1,3)$-invariant Ansatz $u(x) = \varphi(x_\mu x^\mu)$.

A more interesting solution is obtained by putting

$$\begin{aligned} & A_0 = \tau, \quad A_1 = C\sin(\tau/C), \quad A_2 = C\cos(\tau/C), \quad A_3 = 0, \\ & B_\mu = \dot{A}_\mu, \quad \mu = 0, \dots, 3, \end{aligned}$$

where $C \in \mathbb{R}^1$, $C \neq 0$.

With the chosen A_μ, B_μ formulae (2.1.83) take the form

$$\begin{aligned} & u^2 = [x_1 + C\sin(\tau/C)]^2 + [x_2 + C\cos(\tau/C)]^2 + x_3^2 - (x_0 + \tau)^2, \\ & x_0 + \tau - x_1\cos(\tau/C) + x_2\sin(\tau/C) = 0. \end{aligned}$$

After making some simple algebraic manipulations we find an explicit form of the parametric function τ

$$\tau(x,u) = \pm\{2C(u^2 - x_3^2)^{1/2} + x_a x_a - u^2 - C^2\}^{1/2},$$

whence we conclude that the function $u(x)$ is determined by the formula

$$x_0 + \tau(x,u) = x_1\cos\Big(\tau(x,u)/C\Big) - x_2\sin\Big(\tau(x,u)/C\Big) = 0.$$

This solution is new and cannot be in principle obtained within the framework of the Lie approach.

In a similar way we have constructed other particular solutions of the d'Alembert-Hamilton system (2.1.30) with different N, ε which are listed below

1) $N = 0$, $\varepsilon = 1$

$$u(x) = x_0; \tag{2.1.84}$$

2) $N = 1, \quad \varepsilon = 1$

$$u(x) = \pm(x_0^2 - x_3^2)^{1/2}; \tag{2.1.85}$$

3) $N = 2, \quad \varepsilon = 1$

$$u(x) = \pm(x_0^2 - x_1^2 - x_3^2)^{1/2}; \tag{2.1.86}$$

4) $N = 3, \quad \varepsilon = 1$

$$u(x) = \pm(x_0^2 - x_1^2 - x_2^2 - x_3^2)^{1/2}; \tag{2.1.87}$$

5) $N = 0, \quad \varepsilon = -1$

$$\begin{aligned} &u(x) = x_1 \cos W_1(x_0 + x_3) + x_2 \sin W_1(x_0 + x_3) + W_2(x_0 + x_3), \\ &x_0 + x_1 \sin W_1\Big(u(x) + x_3\Big) + x_2 \cos W_1\Big(u(x) + x_3\Big) \\ &\quad + W_2\Big(u(x) + x_3\Big) = 0; \end{aligned} \tag{2.1.88}$$

6) $N = 1, \quad \varepsilon = -1$

$$u(x) = \pm\Big\{\Big(x_1 + W_1(x_0 + x_3)\Big)^2 + \Big(x_2 + W_2(x_0 + x_3)\Big)^2\Big\}^{1/2}; \tag{2.1.89}$$

7) $N = 2, \quad \varepsilon = -1$

$$\begin{aligned} &\pm u(x) + C = x_0 \sinh(\tau/C) - x_1 \cosh(\tau/C), \\ &\quad \tau = -x_2 \pm \Big\{x_0^2 - x_1^2 + \Big(C \pm u(x)\Big)^2\Big\}^{1/2}; \\ &\pm u(x) - C = x_1 \sin(\tau/C) + x_2 \cos(\tau/C), \\ &\quad \tau = -x_0 \pm \Big\{x_1^2 + x_2^2 - \Big(-C \pm u(x)\Big)^2\Big\}^{1/2}; \\ &x_0 \sinh\tau - x_3 \cosh\tau = 2^{-1/2}\{\pm(-u^2(x) - x_\mu x^\mu)^{1/2} \pm u(x)\}, \\ &\quad \tau = \arcsin\Big\{\Big(\sqrt{2}(x_1^2 + x_2^2)^{1/2}\Big)^{-1}\Big(\pm u(x) \mp (-u^2(x) - x_\mu x^\mu)\Big)^{1/2}\Big\} \\ &\qquad - \arcsin\Big\{x_2(x_1^2 + x_2^2)^{-1/2}\Big\}, \\ &u(x) = \pm(x_1^2 + x_2^2 + x_3^2)^{1/2}; \end{aligned} \tag{2.1.90}$$

8) $N = 3, \quad \varepsilon = -1$

$$\begin{aligned} &\pm\Big(u^2(x) - x_3^2\Big)^{1/2} + C = x_0 \sinh(\tau/C) - x_1 \cosh(\tau/C), \\ &\quad \tau = -x_2 \pm \Big\{x_0^2 - x_1^2 + \Big(C \pm [u^2(x) - x_3^2]^{1/2}\Big)^2\Big\}^{1/2}; \\ &\pm\Big(u^2(x) - x_3^2\Big)^{1/2} - C = x_1 \sin(\tau/C) + x_2 \cos(\tau/C), \\ &\quad \tau = -x_0 \pm \Big\{x_1^2 + x_2^2 - \Big(C \mp [u^2(x) - x_3^2]^{1/2}\Big)^2\Big\}^{1/2}. \end{aligned} \tag{2.1.91}$$

Here $\{W_1, W_2\} \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, C is a real non-zero constant.

4. Conditional symmetry of the nonlinear d'Alembert equation. According to the remark made in the very beginning of the section substitution of the Ansatz (2.1.1), where $u(x)$ is an arbitrary solution of the d'Alembert-Hamilton system (2.1.30), into the nonlinear d'Alembert equation

$$\Box w = F_0(w), \tag{2.1.92}$$

reduces it to an ODE for a function φ.

It occurs that the class of Ansätze obtained in this way is substantially wider that the one obtainable by means of the symmetry reduction.

Indeed, within the framework of the symmetry reduction approach to reduce the nonlinear d'Alembert equation (2.1.92) to an ODE one has to construct Ansätze invariant under the three-parameter subgroups of its symmetry group. It is well-known that, provided F_0 is an arbitrary function, the maximal symmetry group admitted by PDE (2.1.92) is the ten-parameter Poincaré group $P(1,3)$ having the generators

$$P_\mu = \partial^\mu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu. \tag{2.1.93}$$

Furthermore, the general form of mentioned Ansätze is given by the formula (2.1.1), where $u(x)$ is an invariant of some three-parameter subgroup of the group $P(1,3)$. An exhaustive description of the invariants of the Poincaré group having the generators (2.1.93) is obtained in [239]. In particular, it is established that any invariant of a three-parameter subgroup of the group $P(1,3)$ can be reduced by an appropriate transformation from the Poincaré group either to the forms (2.1.84)–(2.1.87) or to the forms

$$x_0 + x_3, \quad x_1 + \theta \ln(x_0 + x_3), \quad x_1 + \theta(x_0 + x_3)^2, \quad x_1^2 + x_2^2, \quad x_1^2 + x_2^2 + x_3^2,$$

where θ is a constant.

But the invariants listed above are very special cases of the formulae (2.1.88)–(2.1.90) which in its turn determine only particular solutions of the d'Alembert-Hamilton system.

Such substantial extension of the class of the Ansätze reducing the nonlinear d'Alembert equation is achieved at the expense of its conditional symmetry.

Consider, as an illustration, the Ansatz

$$w(x) = \varphi\Big(x_1 + \rho(x_0 + x_3)\Big), \tag{2.1.94}$$

where ρ is an arbitrary smooth function, obtained by substitution of the first formula from (2.1.88) with $W_1 = 0,\ W_2 = \rho$ into (2.1.1).

In spite of the fact that the Ansatz (2.1.94) is not Poincaré-invariant, it reduces PDE (2.1.92) to the ODE $-\ddot{\varphi} = F_0(\varphi)$. This phenomenon cannot be in principle understood within the framework of the classical Lie approach because the existence of such Ansätze is a consequence of conditional invariance of the nonlinear d'Alembert equation.

Indeed, the manifold (2.1.94) is invariant under the three-parameter Abelian Lie group with the generators

$$Q_1 = \partial_0 - \partial_3, \quad Q_2 = \partial_0 + \partial_3 - 2\dot{\rho}\partial_1, \quad Q_3 = \partial_2$$

(this fact is established by direct computation). Obviously, the operator Q_2 cannot be represented as a linear combination of the operators $P_\mu,\ J_{\mu\nu}$ with constant coefficients which means that equation (2.1.92) is not invariant under the Lie algebra $A = \langle Q_1,\, Q_2,\, Q_2 \rangle$.

We will prove that PDE (2.1.92) is conditionally-invariant under the algebra A. Acting by the second prolongations of the operators Q_a on (2.1.92) we have

$$\widetilde{Q}_1 L = 0, \quad \widetilde{Q}_2 L = 4\ddot{\rho}\partial_1 Q_1 u, \quad \widetilde{Q}_3 L = 0,$$

where $L = \Box u - F_0(u)$.

Hence it follows that the system of PDEs

$$\Box u = F_0(u), \quad Q_a u = 0, \quad a = 1, 2, 3$$

is invariant under the Lie algebra A, the same as what was to be proved.

All Ansätze obtained by substitution of the formulae for $u(x)$ listed in (2.1.88)–(2.1.91) (with the only exception of the last formula from (2.1.90)) into (2.1.1) correspond to the conditional invariance of the nonlinear d'Alembert equation and give rise to the new (non-Lie) reductions of PDE (2.1.92). Hence it follows, in particular, that the nonlinear d'Alembert equation admits an *infinite* conditional symmetry. It will be shown that the nonlinear Dirac and Yang-Mills equations have the same property (see Chapters 6,7).

2.2. Ansätze for the spinor field

We will apply the results given in Chapter 1 to construct Ansätze (1.5.15) reducing Poincaré-invariant multi-dimensional PDEs for the spinor field to equations having a lower dimension.

According to Theorem 1.5.1 to construct an Ansatz (1.5.15) reducing a given equation to PDE with the less number of independent variables we have (see also [100, 155, 233, 236])

- to obtain operators $Q_1,\ Q_2, \ldots, Q_N$ of the form (1.5.11) satisfying conditions of Theorem 1.5.1;
- to integrate the corresponding system of PDEs (1.5.9).

In the present section we consider the case when operators Q_a form a basis of the N-dimensional real Lie algebra which is a subalgebra of the Lie algebra of the invariance group G of the equation under study.

Let $\Sigma_1,\ \Sigma_2, \ldots, \Sigma_M,\ M \geq N$ be the basis elements of the Lie algebra AG.

Definition 2.2.1. Two sets of operators $\{Q_1,\ Q_2, \ldots, Q_N\}$ and $\{Q'_1,\ Q'_2, \ldots, Q'_N\}$ are called G-conjugate if there exist such real parameters $\theta_1, \ldots, \theta_M$ that

$$\exp\{\theta_i \Sigma_i\}\, Q_j\ \exp\{-\theta_i \Sigma_i\} = Q'_j, \quad j = 1, \ldots, M, \tag{2.2.1}$$

summation over repeated indices being implied.

In other words, sets of operators $\{Q_1,\ Q_2, \ldots, Q_N\}$ and $\{Q'_1,\ Q'_2, \ldots, Q'_N\}$ are G-conjugate if there exists a group transformation from the Lie group G having generators $\Sigma_1,\ \Sigma_2, \ldots, \Sigma_M$ which transforms Q_j into $Q'_j,\ j = 1, \ldots, N$. Two Lie algebras with basis elements $Q_i,\ i = 1, \ldots, N$ and $Q'_i,\ i = 1, \ldots, N$ are called G-conjugate if the sets of the first-order differential operators $\{Q_1, \ldots, Q_N\}$ and $\{Q'_1, \ldots, Q'_N\}$ are G-conjugate. Two Lie transformation groups are called G-conjugate if their Lie algebras are G-conjugate.

It is evident that Ansätze invariant under G-conjugate subgroups of the Lie group G are equivalent in a sense that they can be transformed one into another by a suitable group transformation from the group G. That is why we will consider non-conjugate subgroups (subalgebras).

Since the group generated by operators $Q_1, \ldots, Q_N$ is transformed by (2.2.1) into the group having generators $Q'_1, \ldots, Q'_N$, Definition 2.2.1 introduces some relation on the set of subgroups of the Lie group G. It is not difficult to become convinced of the fact that this relation is the equivalence relation on the set of subgroups of the group G and, consequently, it separates this set into mutually disjoint classes. The problem of complete description of such classes (called the problem of a subgroup classification of the group G) has been solved for many important invariance groups of mathematical and theoretical physics equations [9, 10], [14]–[17], [100, 209, 237, 238, 267]. In

particular, a complete description of non-conjugate subgroups of the Poincaré group $P(1,3)$ [9, 10, 209, 237], extended Poincaré group $\tilde{P}(1,3)$ [14, 100, 238] and conformal group $C(1,3)$ [15, 100] is obtained.

We will construct Ansätze invariant under one- and three-parameter subgroups of the groups $P(1,3)$, $\widetilde{P}(1,3)$, $C(1,3)$.

1. $P(1,3)$-invariant Ansätze [150, 152]. The Lie algebra of the Poincaré group has thirteen $P(1,3)$ non-conjugate one-dimensional subalgebras

$$\begin{aligned}
&A_1 = \langle J_{03}\rangle, \quad A_2 = \langle J_{12}\rangle, \quad A_3 = \langle J_{03} + \alpha J_{12}\rangle,\\
&A_4 = \langle J_{01} - J_{03}\rangle, \quad A_5 = \langle P_0\rangle, \quad A_6 = \langle P_3\rangle,\\
&A_7 = \langle P_0 + P_3\rangle, \quad A_8 = \langle J_{03} + \alpha P_1\rangle,\\
&A_9 = \langle J_{12} + \alpha P_3\rangle, \quad A_{10} = \langle J_{12} + \alpha P_0\rangle,\\
&A_{11} = \langle J_{12} + \alpha(P_0 + P_3)\rangle, \quad A_{12} = \langle J_{01} - J_{13} + \alpha P_3\rangle,\\
&A_{13} = \langle J_{01} - J_{13} + \alpha P_2\rangle,
\end{aligned} \tag{2.2.2}$$

where $\alpha \in \mathbb{R}^1$, $\alpha \neq 0$.

Thus, to construct all inequivalent Ansätze invariant under one-parameter subgroups of the group $P(1,3)$ it suffices to integrate system (1.5.9) for each of the operators listed in (2.2.2). The problem of integrating equations (1.5.9) is substantially simplified by the fact that operators (1.1.22) realize a linear representation of the algebra $AP(1,3)$.

At first, we adduce the Ansätze constructed and then consider an example of integration of equations (1.5.9).

A general form of the Ansatz invariant under the group with generators (2.2.2) is as follows

$$\psi(x) = A(x)\varphi(\omega_1,\, \omega_2,\, \omega_3), \tag{2.2.3}$$

where $\varphi = \varphi(\vec{\omega})$ is a new unknown four-component function. A (4×4)-matrix $A(x)$ and scalar functions $\omega_a = \omega_a(x)$ are determined by the choice of a subalgebra from A_1, $A_2, \ldots, A_{13}$ and are given below

$$\begin{aligned}
1)\ \psi(x) &= \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\,(x_0^2 - x_3^2,\, x_1,\, x_2),\\
2)\ \psi(x) &= \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\,(x_0,\, x_1^2 + x_2^2,\, x_3),\\
3)\ \psi(x) &= \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3) - (1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\\
&\quad \times\varphi\Big(x_0^2 - x_3^2,\, x_1^2 + x_2^2,\, \alpha\ln(x_0 + x_3) + \arctan(x_1/x_2)\Big),\\
4)\ \psi(x) &= \exp\Big\{x_1\Big(2(x_0 + x_3)\Big)^{-1}(\gamma_0 + \gamma_3)\gamma_1\Big\}\varphi\,(x_0 + x_3,\, x_0^2 - x_1^2
\end{aligned}$$

$$
\begin{aligned}
& && -x_3^2,\, x_2),\\
5)\ \psi(x) &=&& \varphi\,(x_1,\, x_2,\, x_3),\\
6)\ \psi(x) &=&& \varphi\,(x_0,\, x_1,\, x_2),\\
7)\ \psi(x) &=&& \varphi\,(x_0 + x_3,\, x_1,\, x_2),\\
8)\ \psi(x) &=&& \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\Big(x_0^2 - x_3^2,\, x_2,\, \alpha\ln(x_0 + x_3)\\
& && -x_1\Big),\\
9)\ \psi(x) &=&& \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\Big(x_0,\, x_1^2 + x_2^2,\, x_3\\
& && +\alpha\arctan(x_1/x_2)\Big),\\
10)\ \psi(x) &=&& \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\Big(x_3,\, x_1^2 + x_2^2,\, x_0\\
& && -\alpha\arctan(x_1/x_2)\Big),\\
11)\ \psi(x) &=&& \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\Big(x_0 + x_3,\, x_1^2 + x_2^2,\, x_0 - x_3\\
& && -2\alpha\arctan(x_1/x_2)\Big),\\
12)\ \psi(x) &=&& \exp\{(1/2\alpha)(x_0 + x_3)(\gamma_0 + \gamma_3)\gamma_1\}\varphi\Big((x_0 + x_3)^2 - 2\alpha x_1,\, x_2,\\
& && (x_0 + x_3)^3 - 3\alpha x_1(x_0 + x_3) + 3\alpha^2 x_0\Big),\\
13)\ \psi(x) &=&& \exp\{x_2(2\alpha)^{-1}(\gamma_0 + \gamma_3)\gamma_1\}\varphi\Big(x_0 + x_3,\, x_0^2 - x_1^2 - x_3^2,\, \alpha x_1\\
& && -(x_0 + x_3)x_2\Big).
\end{aligned}
$$

In the above formulae $\exp\{R\} = \sum\limits_{n=1}^{\infty}(n!)^{-1}R^n + I$, I is the unit (4×4)-matrix.

We will construct the Ansatz invariant under the algebra A_1. Since the operator $Q_1 = J_{03} = -x_0\partial_3 - x_3\partial_0 + (1/2)\gamma_0\gamma_3$ satisfies conditions (1.5.16), the above Ansatz can be looked for in the form (1.5.21) with $n = 4,\ m = 4,\ N = 1$, a (4×4)-matrix $A(x)$ and functions $\omega_1(x),\ \omega_2(x),\ \omega_3(x)$ satisfying equations (1.5.20), (1.5.22). Thus, to construct the Ansatz for the field $\psi(x)$ we have to find a particular solution of the matrix PDE

$$
\Big(x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3\Big)A(x) = 0 \tag{2.2.4}
$$

and to obtain a complete system of functionally-independent first integrals of the PDE

$$(x_0\partial_3 + x_3\partial_0)\omega(x) = 0. \tag{2.2.5}$$

Hereafter, when integrating a matrix PDEs of the type (2.2.4) we use the following identity:

$$\partial_\mu \exp\{Tf(x)\} = \Big(\partial_\mu f(x)\Big) T \exp\{Tf(x)\}, \tag{2.2.6}$$

which holds true for an arbitrary constant (4×4)-matrix T and a smooth scalar function $f(x)$.

We look for a solution of (2.2.4) in the form

$$A(x) = \exp\{\gamma_0\gamma_3 f(x)\}.$$

Substituting the above expression into (2.2.4) and applying (2.2.6) we arrive at the equality

$$\{(x_0\partial_3 + x_3\partial_0)f - 1/2\}\gamma_0\gamma_3 \exp\{\gamma_0\gamma_3 f\} = 0$$

or

$$(x_0\partial_3 + x_3\partial_0)f = 1/2.$$

A particular solution of the above PDE is of the form $f(x) = (1/2)\ln(x_0 + x_3)$, whence it follows that $A(x) = \exp\{(1/2)\ln(x_0 + x_3)\gamma_0\gamma_3\}$.

PDE (2.2.5) is equivalent to the Euler-Lagrange system

$$\frac{dx_0}{x_3} = \frac{dx_1}{0} = \frac{dx_2}{0} = \frac{dx_3}{x_0},$$

whose first integrals can be chosen in the form $\omega_1 = x_0^2 - x_3^2$, $\omega_2 = x_1$, $\omega_3 = x_2$.

Substituting the results obtained into the formula (2.2.3) we obtain an Ansatz invariant under the one-dimensional Lie algebra A_1. The remaining algebras $A_2, \ldots, A_{13}$ are treated in a similar way.

Now we give a complete list of $P(1,3)$ non-conjugate three-dimensional subalgebras of the Lie algebra $AP(1,3)$ following [100, 237]:

$$\begin{aligned}
&A_1 = \langle P_0,\, P_1,\, P_2\rangle, \quad A_2 = \langle P_1,\, P_2,\, P_3\rangle,\\
&A_3 = \langle P_0 + P_3,\, P_1,\, P_2\rangle, \quad A_4 = \langle J_{03},\, P_1,\, P_2\rangle,\\
&A_5 = \langle J_{03},\, P_0 + P_3,\, P_1\rangle, \quad A_6 = \langle J_{03} + \alpha P_2,\, P_0,\, P_3\rangle,\\
&A_7 = \langle J_{03} + \alpha P_2,\, P_0 + P_3,\, P_1\rangle, \quad A_8 = \langle J_{12},\, P_0,\, P_3\rangle,\\
&A_9 = \langle J_{12} + \alpha P_0,\, P_1,\, P_2\rangle, \quad A_{10} = \langle J_{12} + \alpha P_3,\, P_1,\, P_2\rangle,
\end{aligned}$$

$$
\begin{aligned}
&A_{11} = \langle J_{12} + P_0 + P_3,\, P_1,\, P_2\rangle, \quad A_{12} = \langle G_1,\, P_0 + P_3,\, P_2\rangle,\\
&A_{13} = \langle G_1,\, P_0 + P_3,\, P_1 + \alpha P_2\rangle,\\
&A_{14} = \langle G_1 + P_2,\, P_0 + P_3,\, P_1\rangle,\\
&A_{15} = \langle G_1 + P_0,\, P_0 + P_3,\, P_2\rangle,\\
&A_{16} = \langle G_1 + P_0,\, P_1 + \alpha P_2,\, P_0 + P_3\rangle,\\
&A_{17} = \langle J_{03} + \alpha J_{12},\, P_0,\, P_3\rangle, \quad A_{18} = \langle J_{03} + \alpha J_{12},\, P_1,\, P_2\rangle,\\
&A_{19} = \langle J_{12},\, J_{03},\, P_0 + P_3\rangle, \quad A_{20} = \langle G_1,\, G_2,\, P_0 + P_3\rangle,\\
&A_{21} = \langle G_1 + P_2,\, G_2 + \alpha P_1 + \beta P_2,\, P_0 + P_3\rangle,\\
&A_{22} = \langle G_1,\, G_2 + P_1 + \beta P_2,\, P_0 + P_3\rangle,\\
&A_{23} = \langle G_1,\, G_2 + P_2,\, P_0 + P_3\rangle, \quad A_{24} = \langle G_1,\, J_{03},\, P_2\rangle,\\
&A_{25} = \langle J_{03} + \alpha P_1 + \beta P_2,\, G_1,\, P_0 + P_3\rangle,\\
&A_{26} = \langle J_{12} + P_0 + P_3,\, G_1,\, G_2\rangle, \quad A_{27} = \langle J_{03} + \alpha J_{12},\, G_1,\, G_2\rangle,\\
&A_{28} = \langle G_1,\, G_2,\, J_{12}\rangle, \quad A_{29} = \langle J_{01},\, J_{02},\, J_{12}\rangle, \quad A_{30} = \langle J_{12},\, J_{23},\, J_{31}\rangle.
\end{aligned}
\tag{2.2.7}
$$

In (2.2.7) $G_i = J_{0i} - J_{i3},\ i = 1, 2$ and $\langle Q_1,\, Q_2,\, Q_3\rangle$ designates the linear span of operators Q_a.

Ansätze invariant under the algebras (2.2.7) were constructed in [152, 155]. They can be represented in the form

$$\psi(x) = A(x)\varphi(\omega), \tag{2.2.8}$$

where $\varphi(\omega)$ is a new unknown four-component function, a (4×4)-matrix $A(x)$ and scalar function $\omega(x)$ being given below.

$$
\begin{aligned}
1)\ \psi(x) &= \varphi\,(x_3),\\
2)\ \psi(x) &= \varphi\,(x_0),\\
3)\ \psi(x) &= \varphi\,(x_0 + x_3),\\
4)\ \psi(x) &= \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\,(x_0^2 - x_3^2),\\
5)\ \psi(x) &= \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\,(x_2),\\
6)\ \psi(x) &= \exp\{(x_2/2\alpha)\gamma_0\gamma_3\}\varphi\,(x_1),\\
7)\ \psi(x) &= \exp\{(x_2/2\alpha)\gamma_0\gamma_3\}\varphi\left(\alpha \ln(x_0 + x_3) - x_2\right),\\
8)\ \psi(x) &= \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\,(x_1^2 + x_2^2),\\
9)\ \psi(x) &= \exp\{-(x_0/2\alpha)\gamma_1\gamma_2\}\varphi\,(x_3),\\
10)\ \psi(x) &= \exp\{(x_3/2\alpha)\gamma_1\gamma_2\}\varphi\,(x_0),
\end{aligned}
$$

$$
\begin{aligned}
11)\ \psi(x) &= \exp\{(1/4)(x_3 - x_0)\gamma_1\gamma_2\}\varphi(x_0 + x_3), \\
12)\ \psi(x) &= \exp\Big\{\Big(x_1/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\}\varphi(x_0 + x_3), \\
13)\ \psi(x) &= \exp\Big\{\Big((\alpha x_1 - x_2)/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\}\varphi(x_0 + x_3), \\
14)\ \psi(x) &= \exp\{(x_2/2)(\gamma_0 + \gamma_3)\gamma_1\}\varphi(x_0 + x_3), \\
15)\ \psi(x) &= \exp\Big\{-\Big((x_0 + x_3)/2\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\}\varphi\Big(2x_1 + (x_0 + x_3)^2\Big), \\
16)\ \psi(x) &= \exp\Big\{-\Big((x_0 + x_3)/2\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\}\varphi\Big(2(x_2 - \alpha x_1) \\
&\quad -\alpha(x_0 + x_3)^2\Big), \\
17)\ \psi(x) &= \exp\{-(1/2\alpha)(\gamma_0\gamma_3 + \alpha\gamma_1\gamma_2)\arctan(x_1/x_2)\}\varphi(x_1^2 + x_2^2), \\
18)\ \psi(x) &= \exp\{(1/2)(\gamma_0\gamma_3 + \alpha\gamma_1\gamma_2)\ln(x_0 + x_3)\}\varphi(x_0^2 - x_3^2), \\
19)\ \psi(x) &= \exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3) - (1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\} \\
&\quad \times\varphi(x_1^2 + x_2^2), \\
20)\ \psi(x) &= \exp\Big\{\Big(1/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)(\gamma_1 x_1 + \gamma_2 x_2)\Big\}\varphi(x_0 + x_3), \\
21)\ \psi(x) &= \exp\Big\{\Big[2\Big((x_0 + x_3)(x_0 + x_3 + \beta) - \alpha\Big)\Big]^{-1}(\gamma_0 + \gamma_3) \\
&\quad \times\Big[\gamma_1\Big((x_0 + x_3 + \beta)x_1 - \alpha x_2\Big) + \gamma_2\Big((x_0 + x_3)x_2 - x_1\Big)\Big]\Big\} \\
&\quad \times\varphi(x_0 + x_3), \\
22)\ \psi(x) &= \exp\Big\{\Big(2(x_0 + x_3)(x_0 + x_3 + \beta)\Big)^{-1}(\gamma_0 + \gamma_3) \\
&\quad \times\Big[\gamma_1\Big((x_0 + x_3 + \beta)x_1 - x_2\Big) + \gamma_2 x_2(x_0 + x_3)\Big]\Big\}\varphi(x_0 + x_3), \\
23)\ \psi(x) &= \exp\Big\{\Big(2(x_0 + x_3)(x_0 + x_3 + 1)\Big)^{-1}(\gamma_0 + \gamma_3) \\
&\quad \times\Big(\gamma_1 x_1(x_0 + x_3 + 1) + \gamma_2 x_2(x_0 + x_3)\Big)\Big\}\varphi(x_0 + x_3), \\
24)\ \psi(x) &= \exp\Big\{\Big(x_1/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\}\exp\{(1/2)\gamma_0\gamma_3 \\
&\quad \times\ln(x_0 + x_3)\}\varphi(x_0^2 - x_1^2 - x_3^2), \\
25)\ \psi(x) &= \exp\Big\{\Big(1/2(x_0 + x_3)\Big)\Big(x_1 - \alpha\ln(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)\gamma_1\Big\} \\
&\quad \times\exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3)\}\varphi\Big(x_2 - \beta\ln(x_0 + x_3)\Big), \\
26)\ \psi(x) &= \exp\Big\{\Big(1/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)(\gamma_1 x_1 + \gamma_2 x_2)\Big\} \\
&\quad \times\exp\Big(-\Big(1/4(x_0 + x_3)\Big)(x \cdot x)\gamma_1\gamma_2\Big\}\varphi(x_0 + x_3), \\
27)\ \psi(x) &= \exp\Big\{\Big(1/2(x_0 + x_3)\Big)(\gamma_0 + \gamma_3)(\gamma_1 x_1 + \gamma_2 x_2)\Big\}
\end{aligned}
$$

$$\times \exp\{(1/2)(\gamma_0\gamma_3 + \alpha\gamma_1\gamma_2)\ln(x_0 + x_3)\}\varphi\,(x \cdot x).$$

Let us note that triplets of operators Q_a which are basis elements of the algebras A_{28}–A_{30} do not satisfy condition (1.5.10). Consequently, they lead to partially-invariant solutions which are not considered here.

As an example, we will carry out integration of equations (1.5.20), (1.5.22) for the algebra A_4 from (2.2.7). Choosing in (1.5.20), (1.5.22) $n = 4,\ m = 4,\ N = 3,\ Q_1 = -x_0\partial_3 - x_3\partial_0 + (1/2)\gamma_0\gamma_3,\quad Q_2 = \partial_1,\ Q_3 = \partial_2$ yields the following system of PDEs for $A(x),\ \omega(x)$:

$$(x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3)A = 0, \quad \partial_1 A = \partial_2 A = 0, \tag{2.2.9}$$

$$(x_0\partial_3 + x_3\partial_0)\omega = 0, \quad \partial_1\omega = \partial_2\omega = 0. \tag{2.2.10}$$

From the last two equations of system (2.2.9) it follows that $A = A(x_0, x_3)$. Substituting this expression into the first equation we get

$$(x_3\partial_0 + x_0\partial_3 - (1/2)\gamma_0\gamma_3)A(x_0, x_3) = 0,$$

whence

$$A(x) = \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}.$$

It is easy to see that a complete set of functionally-independent first integrals of system (2.2.10) consists of one integral which can be chosen in the form $\omega(x) = x_0^2 - x_3^2$. Thus, we obtain the Ansatz numbered by 4.

2. $\widetilde{P}(1,3)$-invariant Ansätze [148, 150, 152, 155]. Subgroup classification of the extended Poincaré group was carried out in [14, 100, 238]. One-dimensional subalgebras of the algebra $A\widetilde{P}(1,3)$ which are $\widetilde{P}(1,3)$ non-conjugate to subalgebras of the algebra $AP(1,3)$ are equivalent to the following ones:

$$\begin{gathered}\langle J_{01} - J_{13} + \alpha D\rangle, \quad \langle J_{12} + \alpha D\rangle,\\ \langle J_{03} + \beta J_{12} + \alpha D\rangle, \quad \langle J_{03} + \beta J_{12} - D + \alpha P_0\rangle,\end{gathered} \tag{2.2.11}$$

where $\{\alpha, \beta\} \subset \mathbb{R}^1,\ \alpha \neq 0,\ D = x_\mu\partial_\mu + k,\ k \in \mathbb{R}^1$ is the infinitesimal operator of the group of scale transformations (1.1.27).

Ansätze invariant under operators (2.2.11) are given by the formulae

$$\begin{aligned}\psi(x) \;\; = \;\; & (x_0 - x_3)^{-k}\ \exp\{(1/2\alpha)(\gamma_0 + \gamma_3)\gamma_1 \ln(x_0 + x_3)\}\varphi\,(\vec{\omega}),\\ & \omega_1 = (x_0^2 - x_1^2 - x_3^2)x_2^{-2}, \quad \omega_2 = (x_0 + x_3)x_2^{-1},\\ & \omega_3 = \alpha x_1(x_0 + x_3)^{-1} + \ln(x_0 + x_3);\end{aligned}$$

$$\begin{aligned}
\psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{-(1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\}\varphi(\vec{\omega}),\\
&\quad \omega_1 = x_0x_3^{-1},\quad \omega_2 = 2\alpha\arctan(x_1/x_2)-\ln(x_1^2+x_2^2),\\
&\quad \omega_3 = (x_0^2-x_3^2)(x_1^2+x_2^2)^{-1};\\
\psi(x) &= (x_0^2-x_3^2)^{-k/2}\exp\Big\{(1/4)(\gamma_0\gamma_3+\beta\gamma_1\gamma_2)\\
&\quad \times\ln\Big((x_0+x_3)/(x_0-x_3)\Big)\Big\}\varphi(\vec{\omega}),\\
&\quad \omega_1 = (x_0^2-x_3^2)^{\alpha-1}(x_0+x_3)^{2\alpha},\quad \omega_2 = (x_0^2-x_3^2)\\
&\quad \times(x_1^2+x_2^2)^{-1},\quad \omega_3 = \beta\ln(x_1^2+x_2^2)\\
&\quad -2\alpha\arctan(x_1/x_2);\\
\psi(x) &= (2x_0+2x_3-\alpha)^{-k/2}\exp\Big\{(1/4)(\gamma_0\gamma_3+\beta\gamma_1\gamma_2)\\
&\quad \times\ln\Big((x_0+x_3)/(x_0-x_3)\Big)\Big\}\varphi(\vec{\omega}),\\
&\quad \omega_1 = (2x_0+2x_3-\alpha)\exp\{(2/\alpha)(x_0-x_3)\},\\
&\quad \omega_2 = (2x_0+2x_3-\alpha)(x_1^2+x_2^2)^{-1},\\
&\quad \omega_3 = \beta\ln(x_1^2+x_2^2)+2\arctan(x_1/x_2).
\end{aligned} \tag{2.2.12}$$

Here $\varphi = \varphi(\omega_1, \omega_2, \omega_3)$ is an arbitrary four-component function.

Three-dimensional subalgebras of the algebra $A\widetilde{P}(1,3)$ which are $\widetilde{P}(1,3)$ non-conjugate to subalgebras of the Poincaré algebra are as follows

$$\begin{aligned}
&A_1 = \langle -J_{03}+D+P_0+P_3,\ P_1,\ P_2\rangle,\\
&A_2 = \langle -J_{03}+D+P_0+P_3,\ P_0-P_3,\ P_1\rangle,\\
&A_3 = \langle J_{12}+\alpha(-J_{03}+D+P_0+P_3),\ P_1,\ P_2\rangle,\\
&A_4 = \langle -J_{03}+D+P_0+P_3,\ J_{12}+\alpha(P_0+P_3),\ P_0-P_3\rangle,\\
&A_5 = \langle -J_{03}+D,\ J_{12}+P_0+P_3,\ P_0-P_3\rangle,\\
&A_6 = \langle -J_{03}+2D,\ \widetilde{G}_1+P_0+P_3,\ P_0-P_3\rangle,\\
&A_7 = \langle -J_{03}+2D,\ \widetilde{G}_1+P_0+P_3,\ P_2\rangle,\\
&A_8 = \langle -J_{03}+D,\ \widetilde{G}_1-P_2,\ P_0-P_3\rangle,\\
&A_9 = \langle -J_{03}-D+P_0-P_3,\ \widetilde{G}_1,\ P_2\rangle,\\
&A_{10} = \langle -J_{03}+D,\ \widetilde{G}_1,\ \widetilde{G}_2-P_2\rangle,\\
&A_{11} = \langle -J_{03}-D+P_0-P_3,\ \widetilde{G}_1,\ \widetilde{G}_2\rangle,\\
&A_{12} = \langle J_{12}-\alpha(J_{03}+D-P_0+P_3),\ \widetilde{G}_1,\ \widetilde{G}_2\rangle,\\
&A_{13} = \langle J_{12}-\alpha D,\ P_0,\ P_3\rangle,\quad A_{14} = \langle J_{12}-\alpha D,\ P_1,\ P_2\rangle,\\
&A_{15} = \langle J_{03}+\alpha D,\ P_0,\ P_3\rangle,\quad A_{16} = \langle J_{03}+\alpha D,\ P_0-P_3,\ P_1\rangle,
\end{aligned}$$

$$
\begin{aligned}
&A_{17} = \langle J_{03} + \alpha D,\, P_1,\, P_2\rangle, \quad A_{18} = \langle J_{12} - \alpha J_{03} - \beta D,\, P_0,\, P_3\rangle,\\
&A_{19} = \langle J_{12} - \alpha J_{03} - \beta D,\, P_1,\, P_2\rangle,\\
&A_{20} = \langle \widetilde{G}_1 - \alpha D,\, P_0 - P_3,\, P_2\rangle,\\
&A_{21} = \langle \widetilde{G}_1 - \alpha D,\, P_0 - P_3,\, P_1\rangle,\\
&A_{22} = \langle \widetilde{G}_1 - \alpha D,\, P_0 - P_3,\, P_1 + \beta P_2\rangle,\\
&A_{23} = \langle \widetilde{G}_1 - \alpha D,\, \widetilde{G}_2 - \beta D,\, P_0 - P_3\rangle,\\
&A_{24} = \langle \widetilde{G}_1,\, J_{03} + \alpha D,\, P_0 - P_3\rangle,\\
&A_{25} = \langle \widetilde{G}_1,\, J_{03} + \alpha D,\, P_2\rangle,\\
&A_{26} = \langle J_{12} - \alpha D,\, J_{03} + \beta D,\, P_0 - P_3\rangle,\\
&A_{27} = \langle J_{03},\, J_{12},\, D\rangle, \quad A_{28} = \langle P_0,\, P_3,\, D\rangle,\\
&A_{29} = \langle P_1,\, P_2,\, D\rangle, \quad A_{30} = \langle P_0 + P_3,\, P_1,\, D\rangle,\\
&A_{31} = \langle P_0,\, J_{12},\, D\rangle, \quad A_{32} = \langle P_3,\, J_{12},\, D\rangle,\\
&A_{33} = \langle P_0 + P_3,\, J_{12},\, D\rangle, \quad A_{34} = \langle P_1,\, J_{03},\, D\rangle,\\
&A_{35} = \langle P_0 + P_3,\, J_{03},\, D\rangle, \quad A_{36} = \langle P_0 + P_3,\, J_{12} + \alpha J_{03},\, D\rangle,\\
&A_{37} = \langle P_0 + P_3,\, \widetilde{G}_1,\, D\rangle, \quad A_{38} = \langle P_2,\, \widetilde{G}_1,\, D\rangle,\\
&A_{39} = \langle \widetilde{G}_1,\, \widetilde{G}_2,\, D\rangle, \quad A_{40} = \langle \widetilde{G}_1,\, J_{03},\, D\rangle,\\
&A_{41} = \langle P_0 + P_3,\, \widetilde{G}_1,\, \widetilde{G}_2 + D\rangle, \quad A_{42} = \langle \widetilde{G}_1,\, \widetilde{G}_2,\, J_{03} + \alpha D\rangle,\\
&A_{43} = \langle \widetilde{G}_1,\, \widetilde{G}_2,\, J_{12} + \alpha D\rangle, \quad A_{44} = \langle \widetilde{G}_1,\, \widetilde{G}_2,\, J_{12} + \alpha J_{03} + \beta D\rangle,
\end{aligned}
\tag{2.2.13}
$$

where $\widetilde{G}_i = -J_{0i} - J_{i3},\ i = 1,2;\ \{\alpha, \beta\} \subset \mathbb{R}^1$.

Without going into details of integration of equations (1.5.22), (1.5.20) we list the Ansätze for the spinor field $\psi(x)$ invariant under the three-dimensional subalgebras (2.2.13).

$$
\begin{aligned}
1)\ \psi(x) &= (x_0 + x_3)^{-k/2} \exp\{(1/4)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\Big(\ln(x_0 + x_3)\\
&\quad -x_0 + x_3\Big),\\
2)\ \psi(x) &= x_2^{-k} \exp\{(1/2)\gamma_0\gamma_3 \ln x_2\}\varphi\,(x_0 - x_3 - 2\ln x_2),\\
3)\ \psi(x) &= (x_0 + x_3)^{-k/2} \exp\{(1/4\alpha)(\alpha\gamma_0\gamma_3 - \gamma_1\gamma_2)\ln(x_0 + x_3)\}\\
&\quad \times\varphi\Big(\ln(x_0 + x_3) - x_0 + x_3\Big),\\
4)\ \psi(x) &= (x_1^2 + x_2^2)^{-k/2} \exp\{(1/2)\gamma_1\gamma_2 \arctan(x_2/x_1) + (1/4)\gamma_0\gamma_3\\
&\quad \times \ln(x_1^2 + x_2^2)\}\varphi\Big(\alpha \arctan(x_2/x_1) + (1/2)(x_0 - x_3)\\
&\quad -(1/2)\ln(x_1^2 + x_2^2)\Big),
\end{aligned}
$$

$$
\begin{aligned}
5)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)+(1/4)\gamma_0\gamma_3\\
&\quad\times\ln(x_1^2+x_2^2)\}\varphi\left(\arctan(x_2/x_1)+(x_0-x_3)/2\right),\\
6)\ \psi(x) &= x_2^{-k}\exp\{-(1/4)\gamma_1(\gamma_0-\gamma_3)(x_0-x_3)\}\exp\{(1/4)\gamma_0\gamma_3\ln x_2\}\\
&\quad\times\varphi\left(x_2[(x_0-x_3)^2-4x_1]^{-1}\right),\\
7)\ \psi(x) &= \left((x_0-x_3)^2-4x_1\right)^{-k}\exp\{-(1/4)\gamma_1(\gamma_0-\gamma_3)(x_0-x_3)\}\\
&\quad\times\exp\left\{(1/4)\gamma_0\gamma_3\ln\left((x_0-x_3)^2-4x_1\right)\right\}\varphi\left([(x_0-x_3)^2-4x_1]^3\right.\\
&\quad\left.\times[(x_0-x_3)^3-6(x_0-x_3)x_1+6(x_0+x_3)]^{-2}\right),\\
8)\ \psi(x) &= \left(x_1(x_0-x_3)^{-1}-x_2\right)^{-k}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\\
&\quad\times\exp\left\{(1/2)\gamma_0\gamma_3\ln\left(x_1(x_0-x_3)^{-1}-x_2\right)\right\}\varphi\,(x_0-x_3),\\
9)\ \psi(x) &= (x_0-x_3)^{-k/2}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\\
&\quad\times\exp\{-(1/4)\gamma_0\gamma_3\ln(x_0-x_3)\}\varphi\left((x_0^2-x_1^2-x_3^2)(x_0-x_3)^{-1}\right.\\
&\quad\left.+\ln(x_0-x_3)\right),\\
10)\ \psi(x) &= \left(x\cdot x+(x_0^2-x_1^2-x_3^2)(x_0-x_3)^{-1}\right)^{-k/2}\exp\{-(1/2)x_1\\
&\quad\times(x_0-x_3)^{-1}\gamma_1(\gamma_0-\gamma_3)\}\exp\{-(1/2)x_2(x_0-x_3+1)^{-1}\\
&\quad\times\gamma_2(\gamma_0-\gamma_3)\}\exp\left\{(1/4)\gamma_0\gamma_3\ln\left(x\cdot x+(x_0^2-x_1^2-x_3^2)\right.\right.\\
&\quad\left.\left.\times(x_0-x_3)^{-1}\right)\right\}\varphi\,(x_0-x_3),\\
11)\ \psi(x) &= (x_0-x_3)^{-k/2}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\exp\{-(1/4)\gamma_0\gamma_3\ln(x_0-x_3)\}\varphi\left((x\cdot x)(x_0-x_3)^{-1}\right.\\
&\quad\left.+\ln(x_0-x_3)\right),\\
12)\ \psi(x) &= (x_0-x_3)^{-k/2}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\exp\{(1/4\alpha)(\gamma_1\gamma_2-\alpha\gamma_0\gamma_3)\ln(x_0-x_3)\}\varphi\left((x\cdot x)\right.\\
&\quad\left.\times(x_0-x_3)^{-1}+\ln(x_0-x_3)\right),\\
13)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)\}\varphi\left(\alpha\arctan(x_2/x_1)\right.\\
&\quad\left.-(1/2)\ln(x_1^2+x_2^2)\right),\\
14)\ \psi(x) &= x_3^{-k}\exp\{(1/2\alpha)\gamma_1\gamma_2\ln x_3\}\varphi\,(x_0/x_3),
\end{aligned}
$$

$$
\begin{aligned}
15)\ \psi(x) &= x_2^{-k}\exp\{-(1/2\alpha)\gamma_0\gamma_3\ln x_2\}\varphi\,(x_1/x_2),\\
16)\ \psi(x) &= x_2^{-k}\exp\{-(1/2\alpha)\gamma_0\gamma_3\ln x_2\}\varphi\left((x_0-x_3)x_2^{-(\alpha+1)/\alpha}\right),\\
17)\ \psi(x) &= (x_0+x_3)^{k\alpha/2}(x_0-x_3)^{-k\alpha/2}\exp\Big\{(1/4)\gamma_0\gamma_3\ln\Big((x_0+x_3)\\
&\quad\times(x_0-x_3)^{-1}\Big)\Big\}\varphi\left((x_0+x_3)^{(1+\alpha)/2}(x_0-x_3)^{(1-\alpha)/2}\right),\\
18)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{(1/2)(\gamma_1\gamma_2-\alpha\gamma_0\gamma_3)\arctan(x_2/x_1)\}\\
&\quad\times\varphi\left(\beta\arctan(x_2/x_1)-(1/2)\ln(x_1^2+x_2^2)\right),\\
19)\ \psi(x) &= (x_0+x_3)^{k\beta/2\alpha}(x_0-x_3)^{-k\beta/2\alpha}\exp\Big\{(1/4\alpha)(\alpha\gamma_0\gamma_3\\
&\quad-\gamma_1\gamma_2)\ln\Big((x_0+x_3)(x_0-x_3)^{-1}\Big)\Big\}\varphi\left((x_0+x_3)^{(\alpha+\beta)/2}\right.\\
&\quad\left.\times(x_0-x_3)^{(\alpha-\beta)/2}\right),\\
20)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{-(1/2)x_1(x_0-x_3)^{-1}\gamma_1(\gamma_0-\gamma_3)\}\\
&\quad\times\varphi\left(\ln(x_0-x_3)+\alpha x_1(x_0-x_3)^{-1}\right),\\
21)\ \psi(x) &= x_2^{-k}\exp\{(1/2\alpha)\gamma_1(\gamma_0-\gamma_3)\ln(x_0-x_3)\}\varphi\left((x_0-x_3)/x_2\right),\\
22)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{(1/2\beta)\gamma_1(\gamma_0-\gamma_3)(x_2-\beta x_1)(x_0-x_3)^{-1}\}\\
&\quad\times\varphi\left((x_2-\beta x_1)(x_0-x_3)^{-1}-(\beta/\alpha)\ln(x_0-x_3)\right),\\
23)\ \psi(x) &= \exp\Big\{(1/2)\Big(2\alpha k-\gamma_1(\gamma_0-\gamma_3)\Big)x_1(x_0-x_3)^{-1}\Big\}\exp\Big\{(1/2)\\
&\quad\times\Big(2\beta k-\gamma_2(\gamma_0-\gamma_3)\Big)x_2(x_0-x_3)^{-1}\Big\}\varphi\left(\exp\{(\alpha x_1\right.\\
&\quad\left.+\beta x_2)(x_0-x_3)^{-1}\}(x_0-x_3)\right),\\
24)\ \psi(x) &= x_2^{-k}\exp\{-(1/2)x_1(x_0-x_3)^{-1}\gamma_1(\gamma_0-\gamma_3)\}\exp\{-(1/2\alpha)\\
&\quad\times\gamma_0\gamma_3\ln x_2\}\varphi\left((x_0-x_3)x_2^{-(\alpha+1)/\alpha}\right),\\
25)\ \psi(x) &= (x_0^2-x_1^2-x_3^2)^{-k/2}\exp\{-(1/2)x_1(x_0-x_3)^{-1}\gamma_1(\gamma_0-\gamma_3)\}\\
&\quad\times\exp\{-(1/4\alpha)\gamma_0\gamma_3\ln(x_0^2-x_1^2-x_3^2)\}\varphi\left((x_0-x_3)\right.\\
&\quad\left.\times(x_0^2-x_1^2-x_3^2)^{-(\alpha+1)/2\alpha}\right),\\
26)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\Big\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)+(1/4)\gamma_0\gamma_3\\
&\quad\times\Big(\ln(x_1^2+x_2^2)-2\ln(x_0-x_3)\Big)\Big\}\varphi\left((1/2)(\beta+1)\ln(x_1^2+x_2^2)\right.\\
&\quad\left.-\beta\ln(x_0-x_3)-\alpha\arctan(x_2/x_1)\right),
\end{aligned}
$$

$$
\begin{aligned}
27)\ \psi(x) &= (x\cdot x)^{-k/2}\exp\Big\{(1/4)\gamma_0\gamma_3\ln\Big((x_0+x_3)(x_0-x_3)^{-1}\Big)\Big\}\\
&\quad\times\exp\{-(1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\}\varphi\Big((x_0^2-x_3^2)(x_1^2+x_2^2)^{-1}\Big),\\
28)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\varphi\,(x_1/x_2),\\
29)\ \psi(x) &= (x_0^2-x_3^2)^{-k/2}\varphi\,(x_0/x_3),\\
30)\ \psi(x) &= (x_0-x_3)^{-k}\varphi\,\Big(x_2(x_0-x_3)^{-1}\Big),\\
31)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)\}\\
&\quad\times\varphi\,\Big((x_1^2+x_2^2)^{1/2}x_3^{-1}\Big),\\
32)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)\}\\
&\quad\times\varphi\,\Big((x_1^2+x_2^2)^{1/2}x_0^{-1}\Big),\\
33)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\Big\{(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)\}\\
&\quad\times\varphi\,\Big((x_1^2+x_2^2)^{1/2}(x_0-x_3)^{-1}\Big),\\
34)\ \psi(x) &= (x_0^2-x_3^2)^{-k/2}\exp\Big\{(1/4)\gamma_0\gamma_3\ln\Big((x_0+x_3)/(x_0-x_3)\Big)\Big\}\\
&\quad\times\varphi\,\Big((x_0^2-x_3^2)^{1/2}x_2^{-1}\Big),\\
35)\ \psi(x) &= x_1^{-k}\exp\Big\{(1/2)\gamma_0\gamma_3\ln\Big(x_1(x_0-x_3)^{-1}\Big)\Big\}\varphi\,(x_2/x_1),\\
36)\ \psi(x) &= (x_1^2+x_2^2)^{-k/2}\exp\Big\{(1/2)\gamma_0\gamma_3\ln\Big((x_1^2+x_2^2)^{1/2}(x_0-x_3)^{-1}\Big)\\
&\quad+(1/2)\gamma_1\gamma_2\arctan(x_2/x_1)\Big\}\varphi\,\Big(\ln(x_1^2+x_2^2)^{1/2}-\ln(x_0-x_3)\\
&\quad+\alpha\arctan(x_2/x_1)\Big),\\
37)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\\
&\quad\times\varphi\,\Big(x_2(x_0-x_3)^{-1}\Big),\\
38)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\\
&\quad\times\varphi\,\Big((x_0^2-x_1^2-x_3^2)(x_0-x_3)^{-2}\Big),\\
39)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\varphi\,\Big(x\cdot x(x_0-x_3)^{-2}\Big),\\
40)\ \psi(x) &= x_2^{-k}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\exp\{(1/2)\gamma_0\gamma_3\\
&\quad\times\ln\Big(x_2(x_0-x_3)^{-1}\Big)\Big\}\varphi\,\Big((x_0^2-x_1^2-x_3^2)x_2^{-2}\Big),\\
41)\ \psi(x) &= (x_0-x_3)^{-k}\exp\Big\{(1/2)(\gamma_0-\gamma_3)\Big(\gamma_1x_1(x_0-x_3)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -\gamma_2\ln(x_0-x_3)\Big)\Big\}\varphi\left(\ln(x_0-x_3)+x_2(x_0-x_3)^{-1}\right),\\
42)\ \psi(x) &= (x\cdot x)^{-k/2}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\exp\Big\{-(1/4\alpha)\gamma_0\gamma_3\,x\cdot x\Big\}\varphi\left((x\cdot x)^{\alpha+1}(x_0-x_3)^{-2\alpha}\right),\\
43)\ \psi(x) &= (x_0-x_3)^{-k}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\exp\{(1/2\alpha)\gamma_0\gamma_3\ln(x_0-x_3)\}\varphi\left(x\cdot x(x_0-x_3)^{-2}\right),\\
44)\ \psi(x) &= (x\cdot x)^{-k/2}\exp\{(1/2)(x_0-x_3)^{-1}(\gamma_0-\gamma_3)(\gamma_1x_1+\gamma_2x_2)\}\\
&\quad\times\exp\{(1/4\beta)(\gamma_1\gamma_2-\alpha\gamma_0\gamma_3)\ln(x\cdot x)\}\varphi\Big((x\cdot x)^{\alpha+\beta}\\
&\quad\times(x_0-x_3)^{-2\beta}\Big).
\end{aligned}
$$

3. Conformally-invariant Ansätze. Complete classification of $C(1,3)$ non-conjugate subgroups of the conformal group was obtained quite recently [15, 100]. We use this classification to construct Ansätze for the spinor field $\psi(x)$ invariant under one- and three-parameter subgroups of the group $C(1,3)$.

$C(1,3)$ non-conjugate one-parameter subgroups of the conformal group which are not $C(1,3)$-conjugate to subgroups of the group $\widetilde{P}(1,3)$ are generated by the following operators:

$$
\begin{aligned}
&Q_1=Q,\quad Q_2=Q+\varepsilon(P_0-P_3),\\
&Q_3=J_{12}+\alpha Q,\quad Q_4=Q+\alpha(D-J_{03}),\\
&Q_5=\beta J_{12}+\alpha Q+\varepsilon(P_0-P_3),\\
&Q_6=\alpha J_{12}+Q-J_{01}-J_{13}-P_2,\\
&Q_7=\delta J_{12}+\alpha Q+\beta(D-J_{03}),\\
&Q_8=P_0+K_0,\quad Q_9=\alpha(P_0+K_0)+J_{12},\\
&Q_{10}=\alpha(P_0+K_0)+J_{12}+\beta(P_3-K_3),\\
&Q_{11}=J_{12}+\beta(P_3-K_3).
\end{aligned}
\tag{2.2.14}
$$

Here $Q=(1/2)(K_0-K_3+P_0+P_3),\ \{\alpha,\beta\}\subset\mathbb{R}^1,\ \varepsilon=\pm1.$

Operators (2.2.14) unlike generators of the extended Poincaré group $\widetilde{P}(1,3)$ have quadratic dependence on x_μ. That is why the corresponding system (1.5.20), (1.5.22) is nonlinear with respect to the independent variables x_μ (in particular, equations (1.5.20) with Q of the form (2.2.14) lead to a Riccati-type system of ODEs). To avoid a necessity to integrate a nonlinear Riccati-type system of ODEs we will apply the trick used by Dirac when investigating

conformal invariance of equation (1.1.17) [70]. Relying on the well-known fact of isomorphism of Lie algebras of the groups $C(1,3)$ and $O(2,4)$ he obtained a change of variables connecting the transformation group $C(1,3)$ of the form (1.1.24)–(1.1.28) with the group of homogeneous linear transformations of some six-dimensional projective space preserving the quadratic form $z_1^2+z_2^2-z_3^2-z_4^2-z_5^2-z_6^2$. And what is more, generators of the group $O(2,4)$ were linear in the variables $z_A,\ A=1,\dots,6$.

Consider the following representation of the Lie algebra $AO(2,4)$:

$$\begin{aligned}
&\Omega_{1\,2}=z_1\partial_{z_2}-z_2\partial_{z_1}+(i/2)\gamma_4\gamma_0,\\
&\Omega_{1\,2+a}=-z_1\partial_{z_{2+a}}-z_{2+a}\partial_{z_1}+(i/2)\gamma_4\gamma_a,\\
&\Omega_{2\,2+a}=-z_2\partial_{z_{2+a}}-z_{2+a}\partial_{z_2}+(1/2)\gamma_0\gamma_a,\\
&\Omega_{2+a\,2+b}=-z_{2+a}\partial_{z_{2+b}}+z_{2+b}\partial_{z_{2+a}}+(1/2)\gamma_a\gamma_b,\\
&\Omega_{1\,6}=z_1\partial_{z_6}-z_6\partial_{z_1}+(i/2)\gamma_4,\\
&\Omega_{2\,6}=z_2\partial_{z_6}-z_6\partial_{z_2}+(1/2)\gamma_0,\\
&\Omega_{2+a\,6}=-z_{2+a}\partial_{z_6}+z_6\partial_{z_{2+a}}+(1/2)\gamma_a,\quad a,b=1,2,3
\end{aligned}\tag{2.2.15}$$

(the remaining elements of $AO(2,4)$ are obtained by the rule $\Omega_{AB}=-\Omega_{BA}$, $A,B=1,\dots,6,\ A\neq B$).

It is straightforward to verify that operators (2.2.15) do satisfy commutation relations of the algebra $AO(2,4)$

$$[\Omega_{AB},\,\Omega_{CD}]=(\rho_{AD}\Omega_{BC}+\rho_{BC}\Omega_{AD}-\rho_{AC}\Omega_{BD}-\rho_{BD}\Omega_{AC}),$$

where $\rho_{AB}=\mathrm{diag}(1,\ 1,-1,-1,-1,-1)$ is the metric tensor of the pseudo-Euclidean space $R(2,4)$. Next, the isomorphism of the algebras $AO(2,4)$ and $AC(1,3)$ is established by the formulae

$$\begin{aligned}
&P_0=-\Omega_{1\,2}-\Omega_{2\,6},\quad P_a=-\Omega_{1\,2+a}-\Omega_{2+a\,6},\\
&J_{0a}=\Omega_{2\,2+a},\quad J_{ab}=\Omega_{2+a\,2+b},\\
&D=-\Omega_{16},\quad K_0=-\Omega_{1\,2}+\Omega_{2\,6},\\
&K_a=-\Omega_{1\,2+a}+\Omega_{2+a\,6},\quad a,b=1,2,3,\quad a\neq b.
\end{aligned}\tag{2.2.16}$$

The transformation relating the groups $O(2,4)$ and $C(1,3)$ can be represented in the form

$$\begin{aligned}
x_\mu &= z_{\mu+2}\,(z_6-z_1)^{-1},\\
\psi(x) &= (z_6-z_1)^2\Big\{1-(1/2)(z_6-z_1)^{-1}(1+i\gamma_4)\\
&\quad\times(\gamma_0 z_2-\gamma_a z_{2+a})\Big\}\Psi(z),
\end{aligned}\tag{2.2.17}$$

coordinates $z_1, \ldots, z_6$ satisfying an additional constraint

$$z_A z^A \equiv z_1^2 + z_2^2 - z_3^2 - z_4^2 - z_5^2 - z_6^2 = 0.$$

It is important to note that the Lie groups $O(2,4)$ and $C(1,3)$ are not isomorphic. Formulae (2.2.17) determine a projection of the group $O(2,4)$ on the group $C(1,3)$.

On rewriting operators (2.2.16) in the variables x, $\psi(x)$ according to (2.2.17) we get the following expressions for the generators of the conformal group $C(1,3)$:

$$\begin{aligned}
&P_\mu = \partial^\mu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu},\\
&D = x_\mu \partial_\mu + 3/2 + (1/2)(1 - i\gamma_4),\\
&K_\mu = 2x_\mu D - (x \cdot x)\partial^\mu + 2S_{\mu\nu}x^\nu + (1/2)(1 - i\gamma_4)\gamma_\mu,
\end{aligned} \tag{2.2.18}$$

where $S_{\mu\nu} = (1/4)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$, $\mu, \nu = 0, \ldots, 3$, $\mu < \nu$.

Hence we conclude that an Ansatz invariant under a subgroup of the group $O(2,4)$ with generators (2.2.15) is transformed by (2.2.17) into an Ansatz invariant under a subgroup of the group $C(1,3)$ with generators (2.2.18). But the above arguments cannot be immediately applied to construct conformally-invariant Ansätze reducing the massless Dirac equation (1.1.17). The matter is that on the set of solutions of equation (1.1.17) a representation of the algebra $AC(1,3)$ inequivalent to the representation (2.2.18) is realized (see Section 1.1). To avoid this difficulty we will modify the change of variables (2.2.17). Let us consider the group $O(2,4)$ acting on the space of eight-component spinors $\widetilde{\Psi}$ which depend on six variables $z_1, \ldots, z_6$. Its generators are chosen as follows

$$\begin{aligned}
&\Omega_{1\,2} = z_1\partial_{z_2} - z_2\partial_{z_1} + (1/2)\sigma\Gamma_0,\\
&\Omega_{1\,2+a} = -z_1\partial_{z_{2+a}} - z_{2+a}\partial_{z_1} + (1/2)\sigma\Gamma_a,\\
&\Omega_{2\,2+a} = -z_2\partial_{z_{2+a}} - z_{2+a}\partial_{z_2} + (1/2)\Gamma_0\Gamma_a,\\
&\Omega_{2+a\,2+b} = -z_{2+a}\partial_{z_{2+b}} + z_{2+b}\partial_{z_{2+a}} + (1/2)\Gamma_a\Gamma_b,\\
&\Omega_{1\,6} = z_1\partial_{z_6} - z_6\partial_{z_1} + (1/2)\sigma,\\
&\Omega_{2\,6} = z_2\partial_{z_6} - z_6\partial_{z_2} + (1/2)\Gamma_0,\\
&\Omega_{2+a\,6} = -z_{2+a}\partial_{z_6} + z_6\partial_{z_{2+a}} + (1/2)\Gamma_a,\\
&\Omega_{A\,B} = -\Omega_{BA}, \quad A \neq B.
\end{aligned} \tag{2.2.19}$$

Here Γ_μ, σ are (8×8)-matrices of the form

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Using the relations

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2g_{\mu\nu}I, \quad \Gamma_\mu\sigma = -\sigma\Gamma_\mu,$$

we can become convinced of the fact that operators (2.2.19) do form a basis of the Lie algebra $AO(2,4)$.

The change of variables

$$\begin{aligned} x_\mu &= z_{\mu+2}\,(z_6 - z_1)^{-1}, \\ \tilde{\psi}(x) &= (z_6 - z_1)^2\Big\{1 - (1/2)(1+\sigma) \\ &\quad \times(\Gamma_0 z_2 - \Gamma_a z_{2+a})(z_6 - z_1)^{-1}\Big\}\widetilde{\Psi}(z), \end{aligned} \tag{2.2.20}$$

where $\tilde{\psi}(x)$ is an eight-component spinor, establishes a correspondence between the group $O(2,4)$ having the generators (2.2.19) and the group $C(1,3)$ having the generators

$$\begin{aligned} &\widetilde{P}_\mu = \partial^\mu, \quad \widetilde{J}_{\mu\nu} = x_\mu\partial^\nu - x_\nu\partial^\mu + \widetilde{S}_{\mu\nu}, \\ &\widetilde{D} = x_\mu\partial_\mu + 3/2 + (1/2)(1-\sigma), \\ &\widetilde{K}_\mu = 2x_\mu\widetilde{D} - (x\cdot x)\partial^\mu + 2\widetilde{S}_{\mu\nu}x^\nu + (1/2)(1-\sigma)\Gamma_\mu, \end{aligned} \tag{2.2.21}$$

where $\widetilde{S}_{\mu\nu} = (1/4)[\Gamma_\mu, \Gamma_\nu]$.

Lemma 2.2.1. *Let $\tilde{\psi}(x)$ satisfy the equation*

$$\widetilde{Q}\tilde{\psi}(x) = (\alpha_\mu\widetilde{P}_\mu + \beta_{\mu\nu}\widetilde{J}_{\mu\nu} + \delta\widetilde{D} + \theta_\mu\widetilde{K}_\mu)\tilde{\psi}(x) = 0, \tag{2.2.22}$$

where α_μ, $\beta_{\mu\nu}$, δ, θ_μ are some real parameters. Then, the four-component spinor $\psi = (\tilde{\psi}^0, \tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3)^T$ satisfies the following equation:

$$Q\psi(x) = (\alpha_\mu P_\mu + \beta_{\mu\nu}J_{\mu\nu} + \delta D + \theta_\mu K_\mu)\psi(x) = 0, \tag{2.2.23}$$

the operators $P_\mu, \ldots, K_\mu$ being of the form (1.1.22).

Proof. We represent the eight-component function $\tilde{\psi}(x)$ as follows

$$\tilde{\psi}(x) = (1/2)(1+\sigma)\tilde{\psi}_1(x) + (1/2)(1-\sigma)\tilde{\psi}_2(x).$$

Substitution of the above expression into (2.2.22) yields

$$(1/2)(1+\sigma)\widetilde{Q}\tilde{\psi}_1 + (1/2)(1-\sigma)\widetilde{Q}\tilde{\psi}_2 = 0,$$

whence it follows that

$$(1/2)(1+\sigma)\widetilde{Q}\tilde{\psi}_1 = 0. \qquad (2.2.24)$$

Since

$$\begin{aligned}
&(1/2)(1+\sigma)\widetilde{D}\tilde{\psi}_1(x) = (x_\mu\partial_\mu + 3/2)(1/2)(1+\sigma)\tilde{\psi}_1(x),\\
&(1/2)(1+\sigma)\widetilde{K}_\mu\tilde{\psi}_1(x) = \{2x_\mu(x_\nu\partial_\nu + 3/2) - (x\cdot x)\partial^\mu\\
&\quad +2\widetilde{S}_{\mu\nu}x^\nu\}(1/2)(1+\sigma)\tilde{\psi}_1,\\
&\widetilde{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & 0\\ 0 & S_{\mu\nu}\end{pmatrix},
\end{aligned}$$

we conclude that due to (2.2.24) equality (2.2.23) holds true. $\triangleright$

The above arguments can be summarized in the form of the following algorithm of constructing conformally-invariant Ansätze for the spinor field $\psi(x)$:

- using the isomorphism (2.2.16) we establish the correspondence between $C(1,3)$ non-conjugate subalgebras of the algebra $AC(1,3)$ and $O(2,4)$ non-conjugate subalgebras of the algebra $AO(2,4)$;
- integrating the systems of PDEs (1.5.20), (1.5.22) we construct Ansätze invariant under non-conjugate subalgebras of the algebra $AO(2,4)$ having the basis elements (2.2.19);
- using the change of variables (2.2.20) we rewrite the obtained Ansätze in variables $x,\ \psi(x)$;
- acting on the eight-component spinor $\tilde{\psi}(x)$ by the projector $P = (1/2)\times(1+\sigma)$ we arrive at the conformally-invariant Ansätze for the spinor field $\psi(x)$.

We will realize the above algorithm for the operator Q_2 from (2.2.14), the remaining operators being treated in an analogous way.

Due to (2.2.16), (2.2.19) the operator Q_2 takes the form

$$\begin{aligned}
Q_2 &= z_2\partial_{z_1} - z_1\partial_{z_2} + z_5\partial_{z_6} - z_6\partial_{z_5} + \varepsilon(z_6 - z_1)(\partial_{z_2} + \partial_{z_5})\\
&+\varepsilon(z_2 - z_5)(\partial_{z_1} + \partial_{z_6}) - (1/2)\{\Gamma_3 + \sigma\Gamma_0 + \varepsilon(1+\sigma)(\Gamma_0 - \Gamma_3)\}.
\end{aligned}$$

Consequently, to determine matrix function $A(z)$ and scalar functions $\omega_1(z), \ldots, \omega_5(z)$ it is necessary to integrate the equations

$$Q_2A(z) = 0,$$

$$\Big\{z_2\partial_{z_1} - z_1\partial_{z_2} + z_5\partial_{z_6} - z_6\partial_{z_5} + \varepsilon(z_6 - z_1) \\ \times(\partial_{z_2} + \partial_{z_5}) + \varepsilon(z_2 - z_5)(\partial_{z_1} + \partial_{z_6})\Big\}\omega_i(z) = 0, \tag{2.2.25}$$

where $i = 1, \ldots, 5$.

It is convenient to rewrite (2.2.25) by introducing new independent variables

$$\begin{aligned} &u_1 = (z_1 - z_6)^2 + (z_2 - z_5)^2, \\ &u_2 = 2(z_1 - z_6)(z_2 - z_5), \quad u_3 = z_3, \quad u_4 = z_4, \\ &u_5 = (z_1 - z_6)(z_1 + z_6) + (z_2 - z_5)(z_2 + z_5), \\ &u_6 = (z_1 - z_6)(z_2 + z_5) - (z_2 - z_5)(z_1 + z_6). \end{aligned}$$

As a result, equations (2.2.25) read

$$\begin{aligned} &\Big\{-2(u_1^2 - u_2^2)^{1/2}\partial_{u_2} - 2\varepsilon u_1\partial_{u_6}\Big\}\omega_i(u) = 0, \\ &\Big\{-2(u_1^2 - u_2^2)^{1/2}\partial_{u_2} - 2\varepsilon u_1\partial_{u_6} - (1/2)\Big(\sigma\Gamma_0 + \Gamma_3 \\ &\quad +\varepsilon(1+\sigma)(\Gamma_0 - \Gamma_3)\Big)\Big\}A(u) = 0. \end{aligned} \tag{2.2.26}$$

The first equation of system (2.2.26) implies that $\omega_1(u), \ldots, \omega_5(u)$ are the first integrals of the following Euler-Lagrange system:

$$\frac{du_1}{0} = \frac{du_2}{-2(u_1^2 - u_2^2)^{1/2}} = \frac{du_3}{0} = \frac{du_4}{0} = \frac{du_5}{0} = \frac{du_6}{-2\varepsilon u_1}.$$

A complete set of functionally-independent first integrals of the above system can be chosen in the form

$$\begin{aligned} &\omega_1 = \arcsin(u_2/u_1) - \varepsilon(u_6/u_1), \quad \omega_2 = u_1, \\ &\omega_3 = u_2, \quad \omega_4 = u_3, \quad \omega_5 = u_4. \end{aligned}$$

Using the identity (2.2.6) we get the following particular solution of the second equation of system (2.2.26):

$$A(u) = \exp\{-(\varepsilon u_6/4u_1)(\sigma\Gamma_0 + \Gamma_3 + \varepsilon(1+\sigma)(\Gamma_0 - \Gamma_3))\}.$$

Rewriting the above expressions in the variables z_A and substituting these into Ansatz (1.5.21) we have

$$\begin{aligned} \widetilde{\Psi}(z) = \exp\Big\{&-(\varepsilon/4)[(z_1 - z_6)(z_2 + z_5) - (z_2 - z_5) \\ &\times(z_1 + z_6)][(z_1 - z_6)^2 + (z_2 - z_5)^2]^{-1} \\ &\times[\Gamma_3 + \sigma\Gamma_0 + (1+\sigma)(\Gamma_0 - \Gamma_3)]\Big\}\tilde{\varphi}(\omega), \end{aligned} \tag{2.2.27}$$

where $\tilde{\varphi}$ is an arbitrary eight-component function of $\omega_1, \ldots, \omega_5$ and

$$\begin{aligned}
&\omega_1 = \arcsin\Big\{2(z_1 - z_6)(z_2 - z_5)\Big((z_1 - z_6)^2 + (z_2 - z_5)^2\Big)^{-1}\Big\} \\
&\quad -\varepsilon\{(z_1 - z_6)(z_2 + z_5) - (z_2 - z_5)(z_1 + z_6)\} \\
&\quad \times\Big((z_1 - z_6)^2 + (z_2 - z_5)^2\Big)^{-1}, \\
&\omega_2 = (z_1 - z_6)^2 + (z_2 - z_5)^2, \quad \omega_3 = z_3, \quad \omega_4 = z_4, \\
&\omega_5 = z_1^2 + z_2^2 - z_3^2 - z_4^2 - z_5^2 - z_6^2.
\end{aligned}$$

In the initial variables x, $\tilde{\psi}(x)$ Ansatz (2.2.27) reads

$$\begin{aligned}
&\tilde{\psi}(x) = x_1^{-2}\Big\{1 - (1/2)(1 + \sigma)\Gamma \cdot x\Big\} \\
&\quad \times \exp\Big\{\tau(x)\Big(\Gamma_3 + \sigma\Gamma_0 + (1 + \sigma)(\Gamma_0 - \Gamma_3)\Big)\Big\}\tilde{\varphi}\,(\omega_1, \omega_2, \omega_3), \\
&\omega_1 = -\arcsin\Big\{2(x_0 - x_3)\Big(1 + (x_0 - x_3)^2\Big)^{-1}\Big\} \\
&\quad +\varepsilon\Big(x_0 + x_3 + (x_0 - x_3)x \cdot x\Big)\Big(1 + (x_0 - x_3)^2\Big)^{-1}, \\
&\omega_2 = \Big(1 + (x_0 - x_3)^2\Big)x_1^{-2}, \quad \omega_3 = (1 + (x_0 - x_3)^2)x_2^{-2}, \\
&\tau(x) = (\varepsilon/4)\Big(x_0 + x_3 + (x_0 - x_3)x \cdot x\Big)\Big(1 + (x_0 - x_3)^2\Big)^{-1}
\end{aligned}$$

(when deriving the above formulae we use the identities $z_1(z_6 - z_1)^{-1} = (1/2)(x \cdot x - 1)$, $z_6(z_6 - z_1)^{-1} = (1/2)(x \cdot x + 1)$ which follow directly from (2.2.20)).

Acting on the Ansatz obtained by the projector $P = (1/2)(1 + \sigma)$ we get

$$\psi(x) = x_1^{-2}\{\cos^2\tau + \gamma_0\gamma_3\sin^2\tau + \gamma \cdot x(\gamma_0 - \gamma_3)\cos\tau\sin\tau\}\varphi(\omega_1, \omega_2, \omega_3),$$

where $\varphi(\omega)$ is a new four-component function, scalar functions $\tau(x)$, $\omega_a(x)$ are determined above.

Below we adduce Ansätze invariant under operators Q_1, $Q_3, \ldots, Q_{11}$.

<u>the operator Q_1</u>

$$\begin{aligned}
&\psi(x) = R(x, \tau, 1)x_1^{-2}\varphi\,(\vec{\omega}), \\
&\omega_1 = [(x \cdot x - 1)^2 + 4x_0^2]x_1^{-2}, \quad \omega_2 = [(x \cdot x - 1)^2 + 4x_0^2]x_2^{-2}, \\
&\omega_3 = \arctan\{(x \cdot x - 1)(2x_0)^{-1}\} - \arctan\{(x \cdot x + 1)(2x_3)^{-1}\}, \\
&\tau = (1/2)\arctan\{(x \cdot x - 1)(2x_0)^{-1}\} + \pi/2;
\end{aligned}$$

the operator Q_3

$$\begin{aligned}
&\psi(x) = R(x,\tau,\alpha)(x_1^2+x_2^2)^{-1}\exp\{-\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),\\
&\omega_1 = [(x\cdot x-1)^2+4x_0^2](x_1^2+x_2^2)^{-1},\\
&\omega_2 = \arctan\{(x\cdot x-1)(2x_0)^{-1}\}-\alpha\arctan(x_1/x_2),\\
&\omega_3 = \arctan\{(x\cdot x-1)(2x_0)^{-1}\}-\arctan\{(x\cdot x+1)(2x_3)^{-1}\},\\
&\tau = (1/2)\arctan(x_1/x_2);
\end{aligned}$$

the operator Q_4

$$\begin{aligned}
&\psi(x) = R(x,\tau,1)x_1^{-2}\exp\{\tau\alpha(1+\gamma_0\gamma_3)\}\varphi(\vec{\omega}),\\
&\omega_1 = \ln\{x_1^2[1+(x_0-x_3)^2]^{-1}\}\\
&\quad -\alpha\arcsin\{2(x_0-x_3)[1+(x_0-x_3)^2]^{-1}\},\\
&\omega_2 = x_1^{-2}[x_0+x_3+(x_0-x_3)x\cdot x],\\
&\omega_3 = x_2^{-2}[x_0+x_3+(x_0-x_3)x\cdot x],\\
&\tau = (1/4\alpha)\ln\{x_1^2[1+(x_0-x_3)^2]^{-1}\};
\end{aligned}$$

the operator Q_5

$$\begin{aligned}
&\psi(x) = R(x,\tau,\alpha)(x_1^2+x_2^2)^{-1}\exp\{-\beta\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),\\
&\omega_1 = [1+(x_0-x_3)]^2(x_1^2+x_2^2)^{-1},\\
&\omega_2 = \varepsilon\arcsin\{2(x_0-x_3)[1+(x_0-x_3)^2]^{-1}\}\\
&\quad +\alpha[x_0+x_3+(x_0-x_3)x\cdot x][1+(x_0-x_3)^2]^{-1},\\
&\omega_3 = -2\varepsilon\arctan(x_1/x_2)+\beta[x_0+x_3+(x_0-x_3)x\cdot x]\\
&\quad \times[1+(x_0-x_3)^2]^{-1},\\
&\tau = (\varepsilon/2)[x_0+x_3+(x_0-x_3)x\cdot x][1+(x_0-x_3)^2]^{-1};
\end{aligned}$$

the operator Q_6

$$\begin{aligned}
\psi(x) = {} & [1+(x_0-x_3)^2]^{-1}\{\cos^2(\tau/2)+\gamma_0\gamma_3\sin^2(\tau/2)\\
&+\sum_{j=1}^{2}(f_j\cos\tau-g_j\sin\tau)\gamma_j(\gamma_0-\gamma_3)+\gamma\cdot x(\gamma_0-\gamma_3)\\
&\times\cos(\tau/2)\sin(\tau/2)\}\exp\{-(\alpha\tau/2)\gamma_1\gamma_2\}\varphi(\vec{\omega}),\quad j=1,2,
\end{aligned}$$

a) under $\alpha=1$

$$f_1 = g_2 = -\tau/2,\quad f_2 = -g_1 = 1,$$

$$\omega_1 = [x_2(x_0 - x_3) - x_1][1 + (x_0 - x_3)^2]^{-1},$$
$$\omega_2 = -\arcsin\{2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1}\}$$
$$+2[x_1(x_0 - x_3) + x_2][1 + (x_0 - x_3)^2]^{-1},$$
$$\omega_3 = \arcsin\{2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1}\}$$
$$+\Big(x_0 + x_3 + (x_0 - x_3)x \cdot x\Big)\Big((x_0 - x_3)x_2 - x_1\Big)^{-1},$$
$$\tau = \arctan(x_0 - x_3);$$

b) under $\alpha \neq 1$

$$f_1 = g_2 = \sin(1 - \alpha)\tau,$$
$$f_2 = -g_1 = [2(\alpha - 1)]^{-1}[2(\alpha - 1)\cos(\alpha - 1)\tau + 1],$$
$$\omega_1 = [2(x_0 - x_3)x_2 - 2x_1 - (1 - \alpha)(x_1^2 + x_2^2)][1 + (x_0 - x_3)^2]^{-1},$$
$$\omega_2 = (\alpha - 1)\Big\{\Big((x_0 - x_3)x_2 - x_1\Big)^2 + \Big((x_0 - x_3)x_1 + x_2\Big)^2\Big\}$$
$$\times\Big(1 + (x_0 - x_3)^2\Big)^{-2} + 2\Big((x_0 - x_3)x_2 - x_1\Big)\Big(1 + (x_0 - x_3)^2\Big)^{-1},$$
$$\omega_3 = 2\arcsin\Big\{\Big[(\alpha - 1)\Big((x_0 - x_3)x_2 - x_1\Big) + 1 + (x_0 - x_3)^2\Big]$$
$$\times\Big\{\Big[(\alpha - 1)\Big((x_0 - x_3)x_2 - x_1\Big) + 1 + (x_0 - x_3)^2\Big]^2$$
$$+(\alpha - 1)^2[(x_0 - x_3)x_1 + x_2]^2\Big\}^{-1/2}\Big\}$$
$$+(\alpha - 1)\arcsin\Big\{2(x_0 - x_3)\Big(1 + (x_0 - x_3)^2\Big)^{-1}\Big\},$$
$$\tau = \arctan(x_0 - x_3);$$

the operator Q_7

$$\psi(x) = R(x, \tau, \alpha)(x_1^2 + x_2^2)^{-1}\exp\{\beta\tau(1 + \gamma_0\gamma_3) - \delta\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),$$
$$\omega_1 = (x_1^2 + x_2^2)^{-1}[x_0 + x_3 + (x_0 - x_3)x \cdot x],$$
$$\omega_2 = \delta\ln\{(x_1^2 + x_2^2)[1 + (x_0 - x_3)^2]^{-1}\} - 2\beta\arcsin\{x_1(x_1^2 + x_2^2)^{-1}\},$$
$$\omega_3 = \alpha\ln\{(x_1^2 + x_2^2)[1 + (x_0 - x_3)^2]^{-1}\}$$
$$-\beta\arcsin\{2(x_0 - x_3)[1 + (x_0 - x_3)^2]^{-1}\},$$
$$\tau = (1/2\delta)\arctan(x_1/x_2);$$

the operator Q_8

$$\psi(x) = x_1^{-2}(\cos\tau - \gamma \cdot x\gamma_0\sin\tau)\varphi(\vec{w}),$$

$$\omega_1 = x_2/x_1, \quad \omega_2 = x_3/x_1, \quad \omega_3 = (1 + x\cdot x)^2(x_1^2 + x_2^2 + x_3^2)^{-1},$$
$$\tau = (1/2)\arctan\{(x\cdot x - 1)(2x_0)^{-1}\} + \pi/2;$$

<u>the operator Q_9</u>

$$\psi(x) = x_3^{-2}(\cos\alpha\tau - \gamma\cdot x\gamma_0\sin\alpha\tau)\exp\{-\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),$$
$$\omega_1 = (x_1^2 + x_2^2)x_3^{-2}, \quad \omega_2 = (x\cdot x + 1)x_3^{-1},$$
$$\omega_3 = \arctan\{(x\cdot x - 1)(2x_0)^{-1}\} - \alpha\arctan(x_1/x_2),$$
$$\tau = (1/2)\arctan\{(x\cdot x - 1)(2x_0)^{-1}\} + \pi/2;$$

<u>the operator Q_{10}</u>

$$\psi(x) = (x_1^2 + x_2^2)^{-1}\Big(\cos\alpha\tau\cos\beta\tau + \gamma_0\gamma_3\sin\alpha\tau\sin\beta\tau$$
$$+\gamma\cdot x(\gamma_0\sin\alpha\tau\cos\alpha\tau - \gamma_3\cos\alpha\tau\sin\alpha\tau)\Big)\exp\{-\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),$$
$$\omega_1 = \alpha\arctan(x_1/x_2) - \arctan\{(x\cdot x - 1)(2x_0)^{-1}\},$$
$$\omega_2 = \beta\arctan(x_1/x_2) - \arctan\{(x\cdot x + 1)(2x_0)^{-1}\},$$
$$\omega_3 = [(x\cdot x - 1)^2 + 4x_0^2](x_1^2 + x_2^2)^{-1},$$
$$\tau = (1/2)\arctan(x_1/x_2);$$

<u>the operator Q_{11}</u>

$$\psi(x) = (x_1^2 + x_2^2)^{-1}(\cos\beta\tau - \gamma\cdot x\gamma_3\sin\beta\tau)\exp\{-\tau\gamma_1\gamma_2\}\varphi(\vec{\omega}),$$
$$\omega_1 = (x_1^2 + x_2^2)x_0^{-2},$$
$$\omega_2 = -\beta\arctan(x_1/x_2) + \arctan\{(x\cdot x + 1)(2x_3)^{-1}\},$$
$$\omega_3 = (x\cdot x - 1)(x_1^2 + x_2^2)^{-1/2}, \quad \tau = (1/2)\arctan(x_1/x_2).$$

In the above formulae we use the following notation:

$$R(x,\tau,\alpha) = \cos^2\alpha\tau + \gamma_0\gamma_3\sin^2\alpha\tau + \gamma\cdot x(\gamma_0 - \gamma_3)\cos\alpha\tau\sin\alpha\tau,$$

$\varphi = \varphi(\vec{\omega})$ is an arbitrary four-component function of $\omega_1,\ \omega_2,\ \omega_3$.

Three-dimensional $C(1,3)$ non-conjugate subalgebras of the conformal algebra which are $C(1,3)$ inequivalent to subalgebras of the algebra $A\widetilde{P}(1,3)$ are as follows

$$A_1 = \langle Q + J_{12},\ -J_{01} - J_{12} - P_2,\ D - J_{03}\rangle,$$
$$A_2 = \langle Q + \alpha J_{12},\ D - J_{03},\ P_0 - P_3\rangle,$$

$$
\begin{aligned}
&A_3 = \langle Q + J_{12} - J_{01} - J_{13} - P_2,\ -J_{02} - J_{23} + P_1,\ P_0 - P_3\rangle,\\
&A_4 = \langle Q + J_{12} + \alpha(D - J_{03}),\ -J_{01} - J_{13} - P_2,\ P_0 - P_3\rangle,\\
&A_5 = \langle J_{12} + \alpha(D - J_{03}),\ Q + \beta(D - J_{03}),\ P_0 - P_3\rangle,\\
&A_6 = \langle J_{03} + D,\ (1/2)(K_0 - K_3),\ (1/2)(P_0 + P_3)\rangle,\\
&A_7 = \langle Q,\ D - J_{03},\ J_{12}\rangle, \quad A_8 = \langle J_{12},\ Q,\ P_0 - P_3\rangle, \qquad (2.2.28)\\
&A_9 = \langle Q + J_{12},\ -J_{01} - J_{13} - P_2, P_0 - P_3\rangle,\\
&A_{10} = \langle J_{12},\ (1/2)(K_0 + P_0),\ (1/2)(P_3 - K_3)\rangle,\\
&A_{11} = \langle J_{23} + (1/2)(P_1 - K_1),\ J_{31} + (1/2)(P_2 - K_2),\ J_{12}\\
&\qquad + (1/2)(P_3 - K_3)\rangle,\\
&A_{12} = \langle J_{12} + (1/2)(P_3 - K_3),\ -J_{03} - (1/2)(P_1 + K_1),\\
&\qquad (1/2)(P_0 - K_0) + (1/2)(P_2 + K_2)\rangle,\\
&A_{13} = \langle \sqrt{3}J_{01} - J_{02} - D,\ \ P_0 + K_0 + 2(K_2 - P_2),\ K_0 - P_0\\
&\qquad - K_2 - P_2 - \sqrt{3}(K_1 + P_1)\rangle, \quad A_{14} = \langle K_0,\ P_0,\ D\rangle.
\end{aligned}
$$

Here $Q = (1/2)(K_0 - K_3 + P_0 + P_3)$, $\{\alpha, \beta\} \subset \mathbb{R}^1$.

The algorithm of constructing conformally-invariant Ansätze formulated above proves to be very efficient when obtaining Ansätze invariant under three-dimensional subalgebras of the algebra $C(1,3)$ listed in (2.2.28) but computations are much more cumbersome. That is why we omit intermediate computations and write down the final result: the Ansätze for the eight-component spinor field $\Psi(x)$ invariant under three-parameter subgroups of the group $C(1,3)$ with generators (2.2.28).

1) $$
\begin{aligned}
\Psi(x) &= \Big(1 + (x_0 - x_3)^2\Big)\Big(x_2(x_0 - x_3) - x_1\Big)^{-2}\\
&\quad \times\Big(1 - (1/2)(1 + \sigma)\Gamma\cdot x\Big)\exp\Big\{(1/2)(\sigma\Gamma_0 + \Gamma_3\\
&\quad -\Gamma_1\Gamma_2)\tau_1 + (1/2)\Big(\Gamma_1(\Gamma_0 - \Gamma_3) + (1 + \sigma)\Gamma_2\Big)\tau_2\Big\}\\
&\quad \times\exp\{(1/2)(\sigma + \Gamma_0\Gamma_3)\tau_3\}\varphi(\omega),\\
\omega &= 2\Big(x_1(x_0 - x_3) + x_2\Big)\Big(x_2(x_0 - x_3) - x_1\Big)^{-1} + \Big(1 + (x_0 - x_3)^2\Big)\\
&\quad \times\Big(x_0 + x_3 + (x_0 - x_3)x\cdot x\Big)\Big(x_2(x_0 - x_3) - x_1\Big)^{-2},\\
\tau_1 &= \arctan(x_0 - x_3),\\
\tau_2 &= (1/2)\Big(x_0 + x_3 + (x_0 - x_3)x\cdot x\Big)\Big(x_2(x_0 - x_3) - x_1)\Big)^{-1},
\end{aligned}
$$

$$\tau_3 = -\ln\Big\{\Big(1+(x_0-x_3)^2\Big)\Big(x_2(x_0-x_3)-x_1\Big)^{-1}\Big\};$$

2) $$\Psi(x) = (x_1^2+x_2^2)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\{(1/2)(\sigma\Gamma_0+\Gamma_3 -\alpha\Gamma_1\Gamma_2)\tau_1\}\exp\{(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_3\} \times\exp\{(1/2)(\sigma+\Gamma_0\Gamma_3)\tau_2\}\varphi(\omega),$$

$$\omega = \arctan(x_1/x_2)-\alpha\arctan(x_0-x_3), \quad \tau_1=\arctan(x_0-x_3),$$

$$\tau_2 = -(1/2)\ln\Big\{\Big(1+(x_0-x_3)^2\Big)(x_1^2+x_2^2)^{-1}\Big\},$$

$$\tau_3 = (1/2)\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)\Big(1+(x_0-x_3)^2\Big)^{-1};$$

3) $$\Psi(x) = \Big(1+(x_0-x_3)^2\Big)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big) \times\exp\Big\{(1/2)\Big(\sigma\Gamma_0+\Gamma_3-\Gamma_1\Gamma_2-\Gamma_1(\Gamma_0-\Gamma_3)-(1+\sigma)\Gamma_2\Big)\tau_1\Big\} \times\exp\{(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_2\}\exp\Big\{(1/2)\Big((1+\sigma)\Gamma_1 -\Gamma_2(\Gamma_0-\Gamma_3)\Big)\tau_3-(1+\sigma)(\Gamma_0-\Gamma_3)\omega\tau_3\Big\}\varphi(\omega),$$

$$\omega = -\arctan(x_0-x_3)+\Big(x_1(x_0-x_3)+x_2\Big)\Big(1+(x_0-x_3)^2\Big)^{-1},$$

$$\tau_1 = \arctan(x_0-x_3),$$

$$\tau_2 = \Big(x_1(x_0-x_3)+x_2\Big)\Big(x_2(x_0-x_3)-x_1\Big)\Big(1+(x_0-x_3)^2\Big)^{-2} +(1/2)\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)\Big(1+(x_0-x_3)^2\Big)^{-1},$$

$$\tau_3 = \Big(x_2(x_0-x_3)-x_1\Big)\Big(1+(x_0-x_3)^2\Big)^{-1};$$

4) $$\Psi(x) = \Big(1+(x_0-x_3)^2\Big)\Big(x_2(x_0-x_3)-x_1\Big)^{-2}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big) \times\exp\Big\{(1/2)\Big(\Gamma_3+\sigma\Gamma_0-\Gamma_1\Gamma_2+\alpha(\sigma+\Gamma_0\Gamma_3)\Big)\tau_1\Big\} \times\exp\Big\{-(1/2)\Big(\Gamma_1(\Gamma_0-\Gamma_3)+(1+\sigma)\Gamma_2\Big)\tau_2\Big\} \times\exp\{(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_3\}\varphi(\omega),$$

$$\omega = \ln\Big\{\Big(1+(x_0-x_3)^2\Big)\Big(x_2(x_0-x_3)-x_1\Big)^{-1}\Big\} +2\alpha\arctan(x_0-x_3), \quad \tau_1=\arctan(x_0-x_3),$$

$$\tau_2 = \Big(x_1(x_0-x_3)+x_2\Big)\Big(1+(x_0-x_3)^2\Big)^{-1} \times\exp\{-\alpha\arctan(x_0-x_3)\},$$

$$\tau_3 = \Big\{\Big(x_1(x_0-x_3)+x_2\Big)\ \Big(x_2(x_0-x_3)-x_1\Big)\Big(1+(x_0-x_3)^2\Big)^{-2}$$

$$+(1/2)\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)\Big(1+(x_0-x_3)^2\Big)^{-1}\Big\}$$
$$\times\exp\{-\alpha\arctan(x_0-x_3)\};$$

5) $$\Psi(x)=(x_1^2+x_2^2)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\Big\{(1/2)\Big(\sigma\Gamma_0+\Gamma_3$$
$$+\beta(\sigma+\Gamma_0\Gamma_3)\Big)\tau_2\Big\}\exp\Big\{(1/2)\Big(\Gamma_1\Gamma_2-\alpha(\sigma+\Gamma_0\Gamma_3)\Big)\tau_1\Big\}$$
$$\times\exp\{(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_3\}\varphi(\omega),$$
$$\omega=2\alpha\arctan(x_2/x_1)-2\beta\arctan(x_0-x_3)$$
$$+\ln\Big\{(x_1^2+x_2^2)\Big(1+(x_0-x_3)^2\Big)^{-1}\Big\},$$
$$\tau_1=\arctan(x_2/x_1),\quad \tau_2=\arctan(x_0-x_3),$$
$$\tau_3=(1/2)\exp\{2\alpha\arctan(x_2/x_1)-2\beta\arctan(x_0-x_3)\}$$
$$\times\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)\Big(1+(x_0-x_3)^2\Big)^{-1};$$

6) $$\Psi(x)=x_1^{-2}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\{(1/2)(1-\sigma)(\Gamma_0-\Gamma_3)\tau_1\}$$
$$\times\exp\{(1/2)(\sigma-\Gamma_0\Gamma_3)\tau_2\}\exp\{(1/2)(1+\sigma)(\Gamma_0+\Gamma_3)\tau_3\}\varphi(\omega),$$
$$\omega=x_1/x_2,\quad \tau_1=(1/2)(x_0+x_3)(x\cdot x)^{-1},\quad \tau_2=\ln\Big((x\cdot x)/x_1\Big),$$
$$\tau_3=(1/2)x_1^2(x_3-x_0)\Big((x_1^2+x_2^2)x\cdot x\Big)^{-1};$$

7) $$\Psi(x)=(x_1^2+x_2^2)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\{(1/2)(\sigma\Gamma_0+\Gamma_3)\tau_1$$
$$+(1/2)(\sigma+\Gamma_0\Gamma_3)\tau_2-(1/2)\Gamma_1\Gamma_2\tau_3\}\varphi(\omega),$$
$$\omega=\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)(x_1^2+x_2^2)^{-1},$$
$$\tau_1=\arctan(x_0-x_3),\quad \tau_2=(1/4)\ln\Big\{\Big((x\cdot x)^2+(x_0+x_3)^2\Big)$$
$$\times\Big(1+(x_0-x_3)^2\Big)^{-1}\Big\},\quad \tau_3=\arctan(x_1/x_2);$$

8) $$\Psi(x)=(x_1^2+x_2^2)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\{-(1/2)\Gamma_1\Gamma_2\tau_1$$
$$+(1/2)(\sigma\Gamma_0+\Gamma_3)\tau_2+(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_3\}\varphi(\omega),$$
$$\omega=(x_1^2+x_2^2)\Big(1+(x_0-x_3)^2\Big)^{-1},$$
$$\tau_1=\arctan(x_1/x_2),\quad \tau_2=\arctan(x_0-x_3),$$
$$\tau_3=(1/2)\Big(x_0+x_3+(x_0-x_3)x\cdot x\Big)\Big(1+(x_0-x_3)^2\Big)^{-1};$$

9) $$\Psi(x)=\Big(1+(x_0-x_3)^2\Big)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)$$
$$\times\exp\Big\{(1/2)(\sigma\Gamma_0+\Gamma_3-\Gamma_1\Gamma_2)\tau_1-(1/2)\Big((1+\sigma)\Gamma_2$$

$$+\Gamma_1(\Gamma_0-\Gamma_3)\Big)\tau_2+(1/2)(1+\sigma)(\Gamma_0-\Gamma_3)\tau_3\Big\}\varphi(\omega),$$

$$\omega=\Big(x_2(x_0-x_3)-x_1\Big)\Big(1+(x_0-x_3)^2\Big)^{-1},$$

$$\tau_1=\arctan(x_0-x_3),\quad \tau_2=\Big(x_1(x_0-x_3)+x_2\Big)\Big(1+(x_0-x_3)^2\Big)^{-1},$$

$$\tau_3=(1/2)(x_0+x_3)-(1/2)x_1^2(x_0-x_3)^{-1}+(1/2)\Big(1-(x_0-x_3)^2\Big)\times$$

$$\times\Big(x_2(x_0-x_3)-x_1\Big)^2(x_0-x_3)^{-1}\Big(1+(x_0-x_3)^2\Big)^{-2};$$

10) $$\Psi(x)=(x_1^2+x_2^2)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)$$

$$\times\exp\{-(1/2)\Gamma_1\Gamma_2\tau_1+(1/2)\sigma\Gamma_0\tau_2+(1/2)\Gamma_3\tau_3\}\varphi(\omega),$$

$$\omega=\Big(4x_0^2+(x\cdot x-1)^2\Big)(x_1^2+x_2^2)^{-1},$$

$$\tau_1=\arctan(x_1/x_2),\quad \tau_2=\arctan\Big((x\cdot x-1)(2x_0)^{-1}\Big)+\pi/2,$$

$$\tau_3=\arctan\Big((x\cdot x+1)(2x_3)^{-1}\Big);$$

11) $$\Psi(x)=x_0^{-2}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)$$

$$\times(R_1B_1C_1-R_1B_2C_2-R_2B_1C_2-R_2B_2C_1)\varphi(\omega),$$

$$R_1=1+(\tau_1/2)(\Gamma_2\Gamma_3-\Gamma_1)-(\tau_2/2)(\Gamma_3\Gamma_1-\Gamma_2),$$

$$R_2=(\tau_1/2)(\Gamma_3\Gamma_1-\Gamma_2)-(\tau_2/2)(\Gamma_2\Gamma_3-\Gamma_1),$$

$$B_1=(1/2)\Big\{1-\Gamma_1\Gamma_2\Gamma_3+(1/2)\Big(x_1^2+x_2^2+(x\cdot x-1)^2\Big)$$

$$\times(x\cdot x-1)^{-1}(x_1^2+x_2^2)^{-1}\Big(x_1(1+\Gamma_1\Gamma_2\Gamma_3)+x_2(\Gamma_1\Gamma_2-\Gamma_3)\Big)\Big\},$$

$$B_2=(1/4)\Big(x_1^2+x_2^2-(x\cdot x-1)^2\Big)(x\cdot x-1)^{-1}(x_1^2+x_2^2)^{-1}$$

$$\times\Big(x_2(1+\Gamma_1\Gamma_2\Gamma_3)+x_1(\Gamma_3-\Gamma_1\Gamma_2)\Big),$$

$$C_1=1+(\rho_1/2)(\Gamma_2\Gamma_3-\Gamma_1)+(\rho_2/2)(\Gamma_3\Gamma_1-\Gamma_2),$$

$$C_2=(\rho_1/2)(\Gamma_2-\Gamma_3\Gamma_1)+(\rho_2/2)(\Gamma_2\Gamma_3-\Gamma_1),$$

$$\omega=(x\cdot x-1)x_0^{-1},$$

$$\tau_1=(1/4)\Big(2x_2x_3-x_1(x\cdot x+1)\Big)(x_1^2+x_2^2)^{-1},$$

$$\tau_2=(1/4)\Big(2x_1x_3+x_2(x\cdot x+1)\Big)(x_1^2+x_2^2)^{-1},$$

$$\rho_1=-(x\cdot x-1)^2\Big(2x_2x_3+x_1(x\cdot x+1)\Big)(x_1^2+x_2^2)^{-1}$$

$$\times\Big((x\cdot x-1)^2+4x_0^2\Big)^{-1},$$

$$\rho_2 = -(x\cdot x-1)^2\Big(2x_1x_3 - x_2(x\cdot x+1)\Big)(x_1^2+x_2^2)^{-1}\times$$
$$\times\Big((x\cdot x-1)^2+4x_0^2\Big)^{-1};$$

12) $$\Psi(x) = \Big((x\cdot x-1)^2-4x_1^2-4x_2^2\Big)^{-1}\Big(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)$$
$$\times\exp\{(\Gamma_1\Gamma_2-\Gamma_3+\sigma\Gamma_1-\Gamma_0\Gamma_3)\tau_1\}\exp\{(\Gamma_0+\sigma\Gamma_2)\tau_2\}$$
$$\times\exp\Big\{(1/2)(\Gamma_1\Gamma_2-\Gamma_3+\sigma\Gamma_1+\Gamma_0\Gamma_3)\arcsin\Big((2\tau_4-\omega)$$
$$\times(\omega^2+4)^{-1/2}\Big)\Big\}\varphi(\omega),$$
$$\omega=\tau_4^{-1}(\tau_3^2+\tau_4^2-1),\quad \tau_1=x_1(x\cdot x-1-2x_2)^{-1},$$
$$\tau_2=-(1/2)\ln\Big\{(1/2)(x\cdot x-1-2x_2)\Big((x\cdot x-1)^2-4x_1^2-4x_2^2\Big)^{-1/2}\Big\},$$
$$\tau_3=2\Big(x_3(x\cdot x-1-2x_2)-x_1(x\cdot x+1-2x_0)\Big)$$
$$\times(x\cdot x-1-2x_2)^{-1}\Big((x\cdot x-1)^2-4x_1^2-4x_2^2\Big)^{-1/2},$$
$$\tau_4=(2x_0-x\cdot x-1)(x\cdot x-1-2x_2)^{-1};$$

13) $$\Psi(x) = x_3^{-2}(1-(1/2)(1+\sigma)\Gamma\cdot x\Big)\exp\{\tau_1 q_+\}\exp\{\tau_2 q\}$$
$$\times\exp\{\tau_3 q_+\}\exp\{\tau_4 q\}\exp\{\tau_5 q_-\}\varphi(\omega),$$
$$q_\pm=(1/2)(\sigma-\Gamma_0\Gamma_2+\sqrt{3}\Gamma_0\Gamma_1)\pm(1/2)(2\Gamma_2-\sigma\Gamma_0),$$
$$q=(1/2)(\gamma_0+\sigma\Gamma_2+\sqrt{3}\sigma\Gamma_1),$$
$$\tau_1=(1/2)(x\cdot x+1+2x_0)(x\cdot x-1+x_2-\sqrt{3}x_1)^{-1},$$
$$\tau_2=(1/2)\ln\{2(x\cdot x-1+x_2+\sqrt{3}x_1)(x\cdot x-1)^{-1}\},$$

functions $\omega(y_1,y_2)$, $\tau_k(y_1,y_2)$, $k=3,4,5$ being determined by the following relations:

$$Q\tau_3-\tau_3^2+3y_1=0,\quad Q\tau_4+2\tau_3=0,$$
$$Q\tau_5+\exp\{-\tau_4\}=0,\quad \omega=y_2^2-4y_1(y_1^2+1),$$

where

$$Q=2y_2\partial_{y_1}+4(3y_1^2+1)\partial_{y_2},$$
$$y_1=(\sqrt{3}x_2-x_1)(\sqrt{3}x_3)^{-1}+(1/4)(2x_0+x\cdot x+1)^2$$
$$\times x_3^{-1}(x\cdot x-1+x_2+\sqrt{3}x_1)^{-1},$$
$$y_2=(1/4)x_3^{-3/2}(x\cdot x-1+x_2+\sqrt{3}x_1)^{-3/2}$$
$$\times\{(2x_0+x\cdot x+1)^3+2\sqrt{3}(2x_0+x\cdot x+1)(\sqrt{3}x_2-x_1)$$
$$\times(x\cdot x-1+x_2+\sqrt{3}x_1)+2(2x_0-x\cdot x-1)(x\cdot x-1+x_2+\sqrt{3}x_1)^2\}.$$

Basis operators of the algebra A_{14} do not satisfy condition (1.5.10). Consequently, they give rise to a partially-invariant solution which is not considered here.

In the above formulae $\varphi(\omega)$ is an arbitrary eight-component complex-valued function.

To obtain conformally-invariant Ansätze for the four-component Dirac field we act with the projector $P = (1/2)(1+\sigma)$ on expressions 1–13. As a result, we have

$$
\begin{aligned}
1)\ \psi(x) &= [1+(x_0-x_3)^2][x_2(x_0-x_3)-x_1]^{-2}R(\tau_1)\\
&\quad\times\exp\{-(\tau_1/2)\gamma_1\gamma_2\}\exp\{-(\tau_2/2)\gamma_1(\gamma_0-\gamma_3)\}\\
&\quad\times\exp\{(\tau_3/2)(1+\gamma_0\gamma_3)\}\varphi(\omega),\\
2)\ \psi(x) &= (x_1^2+x_2^2)^{-1}R(\tau_1)\exp\{-(\alpha\tau_1/2)\gamma_1\gamma_2\}\\
&\quad\times\exp\{(\tau_2/2)(1+\gamma_0\gamma_3)\}\varphi(\omega),\\
3)\ \psi(x) &= [1+(x_0-x_3)^2]^{-1}R(\tau_1)\exp\{-(\tau_1/2)\gamma_1\gamma_2\}\\
&\quad\times\exp\{-(1/2)(\gamma_1\tau_1+\gamma_2\tau_3)(\gamma_0-\gamma_3)\}\varphi(\omega),\\
4)\ \psi(x) &= [1+(x_0-x_3)^2][x_2(x_0-x_3)-x_1]^{-2}R(\tau_1)\\
&\quad\times\exp\{-(\tau_1/2)\gamma_1\gamma_2+(\alpha\tau_1/2)(1+\gamma_0\gamma_3)\}\\
&\quad\times\exp\{-(\tau_2/2)\gamma_1(\gamma_0-\gamma_3)\}\varphi(\omega),\\
5)\ \psi(x) &= (x_1^2+x_2^2)^{-1}R(\tau_2)\exp\{(\beta\tau_2/2)(1+\gamma_0\gamma_3)\}\\
&\quad\times\exp\{(\tau_1/2)[\gamma_1\gamma_2-\alpha(1+\gamma_0\gamma_3)]\}\varphi(\omega),\\
6)\ \psi(x) &= (x\cdot x)^{-2}(\gamma\cdot x)\exp\{(1/2)(3-\gamma_0\gamma_3)\ln[(x\cdot x)/x_1]\}\varphi(\omega),\\
7)\ \psi(x) &= (x_1^2+x_2^2)^{-1}R(\tau_1)\exp\{(\tau_2/2)(1+\gamma_0\gamma_3)-(\tau_3/2)\gamma_1\gamma_2\}\varphi(\omega),\\
8)\ \psi(x) &= (x_1^2+x_2^2)^{-1}R(\tau_2)\exp\{-(\tau_1/2)\gamma_1\gamma_2\}\varphi(\omega),\\
9)\ \psi(x) &= [1+(x_0-x_3)^2]^{-1}R(\tau_1)\exp\{-(\tau_1/2)\gamma_1\gamma_2\}\\
&\quad\times\exp\{-(\tau_2/2)\gamma_1(\gamma_0-\gamma_3)\}\varphi(\omega),\\
10)\ \psi(x) &= \{x_1^2+x_2^2)^{-1}(\cos(\tau_2/2)\cos(\tau_3/2)+\gamma_0\gamma_3\sin(\tau_2/2)\sin(\tau_3/2)\\
&\quad+\gamma\cdot x[\gamma_0\sin(\tau_2/2)\cos(\tau_3/2)-\gamma_3\cos(\tau_2/2)\sin(\tau_3/2)]\}\\
&\quad\times\exp\{-(\tau_1/2)\gamma_1\gamma_2\}\varphi(\omega).
\end{aligned}
\tag{2.2.29}
$$

Ansätze invariant under the algebras $A_{11},\ A_{12},\ A_{13}$ are given by very cumbersome formulae. Therefore they are not adduced here.

In (2.2.29) $\varphi(\omega)$ is an arbitrary four-component function; $\omega,\ \tau_1,\ \tau_2,\ \tau_3$ are real-valued functions defined above in the formulae 1–13;

$$R(\tau) = \cos^2(\tau/2) + \gamma_0\gamma_3\sin^2(\tau/2) + (1/2)\gamma\cdot x(\gamma_0-\gamma_3)\sin\tau.$$

Thus, the problem of construction of Ansätze for the spinor field invariant under the $C(1,3)$ non-conjugate one- and three-parameter subgroups of the conformal group is completely solved. It is important to note that these Ansätze can be applied to reduction of any spinor equation invariant under the groups $P(1,3)$, $\widetilde{P}(1,3)$, $C(1,3)$ in representation (1.1.24)–(1.1.28).

Now we will say a few words about Ansätze reducing Poincaré-invariant equations for particles with arbitrary spins. Suppose that on the set of solutions of the PDE under study a covariant representation of the Poincaré algebra

$$P_\mu = \partial^\mu, \quad J_{\mu\nu} = x_\mu\partial^\nu - x_\nu\partial^\mu + S_{\mu\nu}, \tag{2.2.30}$$

where $S_{\mu\nu}$ are constant matrices fulfilling the commutation relations of the Lie algebra of the Lorentz group $O(1,3)$, is realized. Then Ansätze invariant under the $P(1,3)$ non-conjugate one- and three-dimensional subalgebras of the algebra with basis elements (2.2.30) are obtained by making in the $P(1,3)$-invariant Ansätze for the spinor field the following replacement:

$$\gamma_0\gamma_a \to 2S_{0a}, \quad \gamma_a\gamma_0 \to -2S_{a0}, \quad \gamma_a\gamma_b \to 2S_{ab}, \quad a \neq b.$$

On applying the same trick to the $\widetilde{P}(1,3)$ Ansätze for the spinor field we get the Ansätze invariant under the $\widetilde{P}(1,3)$ non-conjugate subalgebras of the algebra $A\widetilde{P}(1,3)$ having generators (2.2.30) and $D = x_\mu\partial_\mu + k$ (k may be a constant matrix commuting with $S_{\mu\nu}$).

Another method of constructing Poincaré and conformally-invariant Ansätze for fields with spins $s = 0, 1, 3/2$ via Ansätze for the Dirac field is suggested in Section 2.6.

In conclusion we mention nonlocal Ansätze for the Dirac equation. As established in [153] the real eight-component Dirac equation (1.1.14) admits the Poincaré algebra having the following basis elements:

$$\begin{aligned} P_\mu &= \partial^\mu + \theta(\widetilde{\Gamma}_4 + \widetilde{\Gamma}_5)(\partial^\mu + im\widetilde{\Gamma}_\mu), \\ J_{\mu\nu} &= x_\mu\partial^\nu - x_\nu\partial^\mu + (1/4)(\widetilde{\Gamma}_\mu\widetilde{\Gamma}_\nu - \widetilde{\Gamma}_\nu\widetilde{\Gamma}_\mu). \end{aligned} \tag{2.2.31}$$

Here $\widetilde{\Gamma}_\mu$ are (8×8)-matrices defined in Section 1.1, $\nu = 0,\ldots,3$, $\theta = \text{const}$.

Let us emphasize that the operators P_μ are non-Lie operators because the coefficients of ∂^μ are matrices not proportional to the unit matrix.

We have succeeded in solving systems of PDEs (1.5.20), (1.5.22) for each inequivalent subalgebra of the algebra $AP(1,3)$ listed in (2.2.7). As a result we get $P(1,3)$-inequivalent Ansätze for the spinor field

$$\Psi(x) = A(x)\varphi(\omega),$$

where $A(x)$ is an (8×8)-matrix and $\omega = \omega(x)$ is a scalar function, reducing (1.1.14) to systems of ODEs for $\varphi = \varphi(\omega)$. These Ansätze cannot be, in principle, obtained within the framework of the traditional Lie approach (for more details, see [153]).

2.3. Reduction of Poincaré-invariant spinor equations

According to Consequence 1.5.1, substitution of $P(1,3)$-invariant Ansätze (2.2.3) obtained in the previous section into the Poincaré-invariant equation

$$\{i\gamma_\mu\partial_\mu - f_1 - f_2\gamma_4\}\psi(x) = 0, \tag{2.3.1}$$

where $f_i = f_i(\bar\psi\psi,\, \bar\psi\gamma_4\psi)$, yields a three-dimensional system of PDEs for a four-component function $\varphi = \varphi(\omega_1, \omega_2, \omega_3)$. As a direct computation shows these Ansätze satisfy the relations

$$\begin{aligned}&\bar\psi\psi = \bar\varphi\varphi, \quad \bar\psi\gamma_4\psi = \bar\varphi\gamma_4\varphi,\\ &A^{-1}(x)\{i\gamma_\mu\partial_\mu - f_1 - f_2\gamma_4\}A(x)\varphi(\omega)\\ &= \{\gamma_\mu f_{\mu a}\partial_{\omega_a} + (g_\mu + h_\mu g_4)\gamma_\mu + f_1 + f_2\gamma_4\}\varphi(\omega),\end{aligned} \tag{2.3.2}$$

where $f_i = f_i(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)$; $f_{\mu a},\ g_\mu,\ h_\mu$ are rational functions of $\omega_1,\ \omega_2,\ \omega_3$.

Omitting intermediate computations we adduce a final result: reduced equations for four-component functions $\varphi(\vec\omega)$

$$\begin{aligned}
1)\quad & (1/2)(\gamma_0+\gamma_3)\varphi + \Big(\omega_1(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3\Big)\varphi_{\omega_1} + \gamma_1\varphi_{\omega_2}\\
& +\gamma_2\varphi_{\omega_3} = R,\\
2)\quad & (1/2)\omega_2^{-1/2}\gamma_2\varphi + \gamma_0\varphi_{\omega_1} + 2\omega_2^{1/2}\gamma_2\varphi_{\omega_2} + \gamma_3\varphi_{\omega_3} = R,\\
3)\quad & (1/2)\Big(\gamma_0+\gamma_3+\omega_2^{-1/2}\gamma_2\Big)\varphi + \Big(\omega_1(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\varphi_{\omega_1}\\
& +2\omega_2^{1/2}\gamma_2\varphi_{\omega_2} + \Big(\alpha(\gamma_0+\gamma_3)+\omega_2^{-1/2}\gamma_1\Big)\varphi_{\omega_3} = R,\\
4)\quad & (1/2\omega_1)(\gamma_0+\gamma_3)\varphi + (\gamma_0+\gamma_3)\varphi_{\omega_1} + \Big(\omega_1(\gamma_0-\gamma_3)\\
& +(\omega_2/\omega_1)(\gamma_0+\gamma_3)\Big)\varphi_{\omega_2} + \gamma_2\varphi_{\omega_3} = R,\\
5)\quad & \gamma_1\varphi_{\omega_1} + \gamma_2\varphi_{\omega_2} + \gamma_3\varphi_{\omega_3} = R,\\
6)\quad & \gamma_0\varphi_{\omega_1} + \gamma_1\varphi_{\omega_2} + \gamma_2\varphi_{\omega_3} = R,\\
7)\quad & (\gamma_0+\gamma_3)\varphi_{\omega_1} + \gamma_1\varphi_{\omega_2} + \gamma_2\varphi_{\omega_3} = R,
\end{aligned} \tag{2.3.3}$$

8) $(1/2)(\gamma_0+\gamma_3)\varphi + \Big(\omega_1(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\varphi_{\omega_1} + \gamma_2\varphi_{\omega_2}$
$+\Big(\alpha(\gamma_0+\gamma_3)-\gamma_1\Big)\varphi_{\omega_3} = R,$

9) $(1/2)\omega_2^{-1/2}\gamma_2\varphi + \gamma_0\varphi_{\omega_1} + 2\omega_2^{1/2}\gamma_2\varphi_{\omega_2} + (\gamma_3+\alpha\omega_2^{-1/2}\gamma_1)\varphi_{\omega_3} = R,$

10) $(1/2)\omega_2^{-1/2}\gamma_2\varphi + \gamma_3\varphi_{\omega_1} + 2\omega_2^{1/2}\gamma_2\varphi_{\omega_2} + (\gamma_0-\alpha\omega_2^{-1/2}\gamma_1)\varphi_{\omega_3} = R,$

11) $(1/2)\omega_2^{-1/2}\gamma_2\varphi + (\gamma_0+\gamma_3)\varphi_{\omega_1} + 2\omega_2^{1/2}\gamma_2\varphi_{\omega_2} +$
$+(\gamma_0-\gamma_3-2\alpha\omega_2^{-1/2}\gamma_1)\varphi_{\omega_3} = R,$

12) $-2\alpha\gamma_1\varphi_{\omega_1} + \gamma_2\varphi_{\omega_2} + (3/2)\Big(2\alpha^2\gamma_0+\omega_1(\gamma_0+\gamma_3)\Big)\varphi_{\omega_3} = R,$

13) $(2\alpha)^{-1}\gamma_4(\gamma_0+\gamma_3)\varphi + (\gamma_0+\gamma_3)\varphi_{\omega_1} + \Big(\omega_1(\gamma_0+\gamma_3) - 2\alpha^{-1}\omega_3\gamma_1$
$+(\omega_2+\alpha^{-2}\omega_3^2)\omega_1^{-1}(\gamma_0+\gamma_3)\Big)\varphi_{\omega_2} + (\alpha\gamma_1-\omega_1\gamma_2)\varphi_{\omega_3} = R.$

In (2.3.3) $\varphi_{\omega_a} = \partial_{\omega_a}\varphi$, $R = -if_1(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\varphi - if_2(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\gamma_4\varphi$.

$P(1,3)$-invariant Ansätze (2.2.8) also satisfy conditions of the form (2.3.2)

$$\begin{aligned} &\bar\psi\psi = \bar\varphi\varphi, \quad \bar\psi\gamma_4\psi = \bar\varphi\gamma_4\varphi,\\ &A^{-1}(x)(i\gamma_\mu\partial_\mu - f_1 - f_2\gamma_4)A(x)\varphi(\omega)\\ &\quad = \Big(\rho_\mu\gamma_\mu\partial_\omega + (g_\mu + h_\mu\gamma_4)\gamma_\mu - f_1 - f_2\gamma_4\Big)\varphi(\omega), \end{aligned} \tag{2.3.4}$$

where $f_i = f_i(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)$; ρ_μ, γ_μ, h_μ are rational functions of ω. Using this result we get the following set of reduced equations for four-component functions φ:

1) $\gamma_3\dot\varphi = R,$

2) $\gamma_0\dot\varphi = R,$

3) $(\gamma_0+\gamma_3)\dot\varphi = R,$

4) $(1/2)(\gamma_0+\gamma_3)\varphi + \Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\dot\varphi = R,$

5) $(1/2)(\gamma_0+\gamma_3)\varphi + \gamma_2\dot\varphi = R,$

6) $-(1/2\alpha)\gamma_1\gamma_4\varphi + \gamma_1\dot\varphi = R,$

7) $-(1/2\alpha)\gamma_1\gamma_4\varphi + \Big(\alpha\exp\{-\omega/\alpha\}(\gamma_0+\gamma_3)-\gamma_2\Big)\dot\varphi = R,$

8) $(1/2)\omega^{-1/2}\gamma_2\varphi + 2\omega^{1/2}\gamma_2\dot\varphi = R,$

9) $-(1/2\alpha)\gamma_3\gamma_4\varphi + \gamma_3\dot\varphi = R,$

10) $(1/2\alpha)\gamma_0\gamma_4\varphi + \gamma_0\dot\varphi = R,$

11) $(1/4)(\gamma_0-\gamma_3)\gamma_4\varphi + (\gamma_0+\gamma_3)\dot\varphi = R,$

$$
\begin{aligned}
&12)\quad (1/2\omega)(\gamma_0+\gamma_3)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&13)\quad (1/2\alpha\omega)(\alpha+\gamma_4)(\gamma_0+\gamma_3)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&14)\quad (1/2)(\gamma_0+\gamma_3)\gamma_4\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&15)\quad 2\gamma_1\dot{\varphi}=R,\\
&16)\quad 2(\gamma_2-\alpha\gamma_1)\dot{\varphi}=R,\\
&17)\quad (1/2\alpha)\omega^{-1/2}\gamma_2(\alpha-\gamma_4)\varphi+2\omega^{1/2}\gamma_2\dot{\varphi}=R,\\
&18)\quad (1/2)(\gamma_0+\gamma_3)(1+\alpha\gamma_4)\varphi+\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\dot{\varphi}=R,\\
&19)\quad (1/2)(\gamma_0+\gamma_3+\omega^{-1/2}\gamma_2)\varphi+2\omega^{1/2}\gamma_2\dot{\varphi}=R,\\
&20)\quad \omega^{-1}(\gamma_0+\gamma_3)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&21)\quad (1/2)\Big(\omega(\omega+\beta)-\alpha\Big)^{-1}\Big((1-\alpha)\gamma_4+2\omega+\beta\Big)(\gamma_0+\gamma_3)\varphi\\
&\qquad\ +(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&22)\quad (1/2)\Big(\omega(\omega+\beta)\Big)^{-1}(2\omega+\beta-\gamma_4)(\gamma_0+\gamma_3)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&23)\quad (1/2)\Big(\omega(\omega+1)\Big)^{-1}(2\omega+1)(\gamma_0+\gamma_3)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&24)\quad (\gamma_0+\gamma_3)\varphi+\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\dot{\varphi}=R,\\
&25)\quad (\gamma_0+\gamma_3)\varphi+\Big(\gamma_2-\beta(\gamma_0+\gamma_3)\Big)\dot{\varphi}=R,\\
&26)\quad \Big(\omega^{-1}(\gamma_0+\gamma_3)+(1/4)(\gamma_0-\gamma_3)\gamma_4\Big)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,\\
&27)\quad (1/2)(\gamma_0+\gamma_3)(3+\alpha\gamma_4)\varphi+\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\dot{\varphi}=R.
\end{aligned}
\tag{2.3.5}
$$

Here $\dot{\varphi}=d\varphi/d\omega$, $\quad R=-if_1(\bar{\varphi}\varphi,\bar{\varphi}\gamma_4\varphi)\varphi-if_2(\bar{\varphi}\varphi,\bar{\varphi}\gamma_4\varphi)\gamma_4\varphi.$

Formulae (2.3.2), (2.3.4) can be applied to reduce the equation

$$\partial_\mu\partial^\mu\psi(x)=0 \tag{2.3.6}$$

by means of $P(1,3)$-invariant Ansätze for the spinor field $\psi(x)$. To this end we make use of the identity

$$\partial_\mu\partial^\mu=\gamma_\mu\partial_\mu A(x)A^{-1}(x)\gamma_\mu\partial_\mu \tag{2.3.7}$$

which holds for each invertible (4×4)-matrix $A(x)$. By force of (2.3.7), we have

$$
\begin{aligned}
A^{-1}(x)\partial_\mu\partial^\mu A(x)\varphi(\vec{\omega})&=A^{-1}(x)\gamma_\mu\partial_\mu A(x)A^{-1}(x)\gamma_\mu\partial_\mu A(x)\varphi(\vec{\omega})\\
&=A^{-1}(x)\gamma_\mu\partial_\mu A(x)\Big\{\gamma_\mu f_{\mu a}(\vec{\omega})\varphi_{\omega_a}+\gamma_\mu\Big(g_\mu(\vec{\omega})+h_\mu(\vec{\omega})\gamma_4\Big)\varphi\Big\}\\
&=A^{-1}(x)\Big\{\gamma_\mu f_{\mu a}(\vec{\omega})\partial_{\omega_a}+\gamma_\mu\Big(g_\mu(\vec{\omega})+h_\mu(\vec{\omega})\gamma_4\Big)\Big\}^2\varphi,
\end{aligned}
$$

the forms of functions $f_{\mu a},\ g_\mu,\ h_\mu$ being determined by (2.3.3).

In the same way we establish that $P(1,3)$-invariant Ansätze (2.2.8) reduce equation (2.3.6) to ODE

$$\Big\{\rho_\mu(\omega)\gamma_\mu\partial_\omega + \Big(g_\mu(\omega) + h_\mu(\omega)\gamma_4\Big)\gamma_\mu\Big\}^2\varphi = 0,$$

where functions $\rho_\mu(\omega),\ g_\mu(\omega),\ h_\mu(\omega)$ are determined by (2.3.5).

Provided reduced equations (2.3.3), (2.3.5) possess nontrivial symmetry, their dimension can also be decreased with the use of Theorem 1.5.1. But direct application of the infinitesimal Lie method to investigation of the symmetry of systems of PDEs with variable coefficients (2.3.3), (2.3.5) is, in many cases, impossible without applying symbolic computation packages [108, 109, 202, 252] (for multi-component systems of PDEs with $n > 2$ independent variables these packages are also of little help).

In the papers [152, 155] we suggested a purely algebraic method of investigation of invariance properties of reduced equations. It is based on the following assertion.

Theorem 2.3.1. *Let G be a Lie invariance group of some PDE and H be a normal divisor in G. Then an equation obtained via reduction with the help of an H-invariant Ansatz admits the group G/H (here the symbol / means factorization).*

Proof can be found in [236]. ▷

We use an equivalent formulation of the above theorem: *if there is a PDE admitting a Lie algebra AG whose subalgebra Q is an ideal in AG, then an equation obtained by reduction with the help of a Q-invariant Ansatz is invariant under the Lie algebra AG/Q.*

To apply Theorem 2.3.1 to algebras (2.2.2), (2.2.7) we have have to select the maximal subalgebras of the algebra $AP(1,3)$ such that algebras (2.2.2), (2.2.7) are ideals in these.

From the general theory of Lie algebras (see, e.g. [19, 194, 236]) it follows that the algebra $A\widetilde{G} = \langle Q_1, \ldots, Q_N\rangle$ is an ideal in the Lie algebra $AG = \langle\Sigma_1, \Sigma_2, \ldots, \Sigma_M\rangle$, $M \geq N$ iff

$$[Q_i,\, \Sigma_j] = \lambda_{ij}^k Q_k, \quad \lambda_{ij}^k = \text{const}, \tag{2.3.8}$$

the summation over repeated indices being implied.

Given an explicit form of the elements Q_i, we compute with the aid of (2.3.8) the maximal subalgebra of the algebra AG such that the algebra $A\widetilde{G}$ is an ideal in it. Next, we compute a factor-algebra whose basis elements

according to Theorem 2.5.1 generate an invariance group of the corresponding reduced equation.

The above scheme will be realized for the algebra A_5 from (2.2.2). Substituting $Q = P_0$ into (2.3.8) and putting $N = 1$ we arrive at the following relations for $\Sigma_j = \theta_j^{\mu\nu} J_{\mu\nu} + \theta_j^{\mu} P_{\mu}$:

$$[P_0,\, \theta_j^{\mu\nu} J_{\mu\nu} + \theta_j^{\mu} P_{\mu}] = \lambda_j P_0, \quad j = 1, \dots, M. \tag{2.3.9}$$

Computing the commutators and equating coefficients of the linearly-independent operators $P_\mu,\ J_{\mu\nu}$ yield the system of linear algebraic equations for constants $\theta_j^{\mu\nu},\ \theta_j^{\mu}$

$$\theta_j^{0a} = \theta_j^{a0} = 0, \quad a = 1, 2, 3,$$

$\theta_j^{ab},\ \theta_j^{\mu}$ are arbitrary real constants.

Consequently, the basis of a maximal subalgebra of the algebra $AP(1,3)$ containing the algebra $A_5 = \langle P_0 \rangle$ as an ideal consists of the operators

$$P_\mu, \quad J_{12}, \quad J_{23}, \quad J_{31}. \tag{2.3.10}$$

The basis of the factor-algebra $\langle P_\mu, J_{ab} \rangle / \langle P_0 \rangle$ is formed by those operators from (2.3.10) which are linearly independent of P_0. As a result, we come to the Lie algebra

$$\widetilde{A}_5 = \langle P_1,\, P_2,\, P_3,\, J_{12},\, J_{23},\, J_{31} \rangle \tag{2.3.11}$$

which, according to Theorem 2.3.1, is the invariance algebra of the system 5 from (2.3.3). The explicit form of symmetry operators is obtained by passing from the "old " variables $x,\ \psi(x)$ to the "new" ones $\omega,\ \varphi(\omega)$ according to formula (2.2.3).

Below we write down the invariance algebras of equations (2.3.3)

$$\begin{aligned}
&\widetilde{A}_1 = \langle -\omega_2 \partial_{\omega_3} + \omega_3 \partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\, \partial_{\omega_2},\, \partial_{\omega_3} \rangle,\\
&\widetilde{A}_2 = \langle \omega_3 \partial_{\omega_1} + \omega_1 \partial_{\omega_3} - (1/2)\gamma_0\gamma_3,\, \partial_{\omega_1},\, \partial_{\omega_3} \rangle, \quad \widetilde{A}_3 = \langle \partial_{\omega_3} \rangle,\\
&\widetilde{A}_4 = \langle 2\omega_1\omega_3 \partial_{\omega_2} + \omega_1 \partial_{\omega_3} - (1/2)(\gamma_0 + \gamma_3)\gamma_1, -\omega_1 \partial_{\omega_1} + (1/2)\gamma_0\gamma_3,\\
&\qquad \omega_1 \partial_{\omega_2},\, \partial_{\omega_3} \rangle,\\
&\widetilde{A}_5 = \langle -\omega_1 \partial_{\omega_2} + \omega_2 \partial_{\omega_1} + (1/2)\gamma_1\gamma_2,\, -\omega_2 \partial_{\omega_3} + \omega_3 \partial_{\omega_2} + (1/2)\gamma_2\gamma_3,\\
&\qquad - \omega_3 \partial_{\omega_1} + \omega_1 \partial_{\omega_3} + (1/2)\gamma_3\gamma_1,\, \partial_{\omega_1},\, \partial_{\omega_2},\, \partial_{\omega_3} \rangle,\\
&\widetilde{A}_6 = \langle \omega_1 \partial_{\omega_2} + \omega_2 \partial_{\omega_1} - (1/2)\gamma_0\gamma_1,\, \omega_1 \partial_{\omega_3} + \omega_3 \partial_{\omega_1} - (1/2)\gamma_0\gamma_2,\\
&\qquad - \omega_2 \partial_{\omega_3} + \omega_3 \partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\, \partial_{\omega_1},\, \partial_{\omega_2},\, \partial_{\omega_3} \rangle, \qquad\qquad (2.3.12)\\
&\widetilde{A}_7 = \langle \omega_1 \partial_{\omega_2} - (1/2)(\gamma_0 + \gamma_3)\gamma_1,\, \omega_1 \partial_{\omega_3} - (1/2)(\gamma_0 + \gamma_3)\gamma_2,\, -\omega_1 \partial_{\omega_1}
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1/2)\gamma_0\gamma_3,\ -\omega_2\partial_{\omega_3} + \omega_3\partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\ \partial_{\omega_1},\ \partial_{\omega_2},\ \partial_{\omega_3}\rangle,\\
&\widetilde{A}_8 = \langle\partial_{\omega_2},\ \partial_{\omega_3}\rangle, \quad \widetilde{A}_9 = \langle\partial_{\omega_1},\ \partial_{\omega_3}\rangle, \quad \widetilde{A}_{10} = \langle\partial_{\omega_1},\ \partial_{\omega_3}\rangle,\\
&\widetilde{A}_{11} = \langle\partial_{\omega_1},\ \partial_{\omega_3}\rangle, \quad \widetilde{A}_{12} = \langle\partial_{\omega_2},\ \partial_{\omega_3}\rangle,\\
&\widetilde{A}_{13} = \langle 2\alpha\omega_1\partial_{\omega_3} - (\gamma_0+\gamma_3)\gamma_1,\ 2\omega_3\partial_{\omega_2} + (\omega_1^2-\alpha^2)\partial_{\omega_3}\\
&\qquad + (1/2\alpha)(\gamma_0+\gamma_3)(\alpha\gamma_2 - \gamma_1\omega_1),\ \omega_1\partial_{\omega_2}\rangle.
\end{aligned}
$$

The invariance algebras of systems of ODEs listed in (2.3.5) are obtained in a similar way

$$
\begin{aligned}
&\widetilde{A}_1 = \langle\partial_\omega,\ \gamma_0\gamma_1,\ \gamma_0\gamma_2,\ \gamma_1\gamma_2\rangle, \quad \widetilde{A}_2 = \langle\partial_\omega,\ \gamma_1\gamma_2,\ \gamma_2\gamma_3,\ \gamma_3\gamma_1\rangle,\\
&\widetilde{A}_3 = \langle\gamma_1(\gamma_0+\gamma_3),\ \gamma_2(\gamma_0+\gamma_3),\ \gamma_1\gamma_2,\ \omega\partial_\omega - (1/2)\gamma_0\gamma_3,\ \partial_\omega\rangle,\\
&\widetilde{A}_4 = \langle\gamma_1\gamma_2\rangle, \quad \widetilde{A}_5 = \langle\partial_\omega\rangle, \quad \widetilde{A}_6 = \langle\partial_\omega,\ \gamma_0\gamma_3\rangle, \quad \widetilde{A}_7 = \langle 2\alpha\partial_\omega - \gamma_0\gamma_3\rangle,\\
&\widetilde{A}_8 = \langle\gamma_0\gamma_3\rangle, \quad \widetilde{A}_9 = \langle\partial_\omega,\ \gamma_1\gamma_2\rangle, \quad \widetilde{A}_{10} = \langle\partial_\omega,\ \gamma_1\gamma_2\rangle, \quad \widetilde{A}_{11} = \langle\partial_\omega,\ \gamma_1\gamma_2\rangle,\\
&\widetilde{A}_{12} = \langle\gamma_2(\gamma_0+\gamma_3),\ \omega^{-1}\gamma_1(\gamma_0+\gamma_3),\ \omega\partial_\omega - (1/2)\gamma_0\gamma_3\rangle,\\
&\widetilde{A}_{13} = \langle(\gamma_1+\alpha\gamma_2)(\gamma_0+\gamma_3),\ \omega^{-1}\gamma_1(\gamma_0+\gamma_3),\ \omega\partial_\omega - (1/2)\gamma_0\gamma_3\rangle,\\
&\widetilde{A}_{14} = \langle\gamma_1(\gamma_0+\gamma_3),\ \gamma_2(\gamma_0+\gamma_3),\ \partial_\omega\rangle, \quad \widetilde{A}_{15} = \langle\partial_\omega,\ \gamma_2(\gamma_0+\gamma_3)\rangle,\\
&\widetilde{A}_{16} = \langle\partial_\omega\rangle, \quad \widetilde{A}_{17} = \langle\gamma_0\gamma_3\rangle, \quad \widetilde{A}_{18} = \langle\gamma_1\gamma_2\rangle, \quad \widetilde{A}_{19} = \oslash, \qquad (2.3.13)\\
&\widetilde{A}_{20} = \langle\omega\partial_\omega - (1/2)\gamma_0\gamma_3,\ \omega^{-1}\gamma_1(\gamma_0+\gamma_3),\ \omega^{-1}\gamma_2(\gamma_0+\gamma_3),\ \gamma_1\gamma_2\rangle,\\
&\widetilde{A}_{21} = \langle[\omega(\omega+\beta)-\alpha]^{-1}(\gamma_0+\gamma_3)[(\omega+\beta)\gamma_1 - \gamma_2],\ [\omega(\omega+\beta)\\
&\qquad - \alpha]^{-1}(\gamma_0+\gamma_3)(\omega\gamma_2 - \alpha\gamma_1)\rangle,\\
&\widetilde{A}_{22} = \langle\omega^{-1}\gamma_1(\gamma_0+\gamma_3),\ [\omega(\omega+\beta)-\alpha]^{-1}(\gamma_0+\gamma_3)(\omega\gamma_2 - \gamma_1)\rangle,\\
&\widetilde{A}_{23} = \langle\omega^{-1}\gamma_1(\gamma_0+\gamma_3),\ (\omega+1)^{-1}\gamma_2(\gamma_0+\gamma_3)\rangle, \quad \widetilde{A}_{24} = \langle\gamma_1\gamma_2\rangle,\\
&\widetilde{A}_{25} = \langle\partial_\omega,\ \gamma_1(\gamma_0+\gamma_3)\rangle, \quad \widetilde{A}_{26} = \langle\gamma_1\gamma_2\rangle, \quad \widetilde{A}_{27} = \langle\gamma_1\gamma_2\rangle.
\end{aligned}
$$

It is worth noting that any Poincaré-invariant spinor PDE after being reduced by means of the $P(1,3)$-invariant Ansätze (2.2.3), (2.2.2) is invariant under Lie algebras (2.3.12), (2.3.13). But for the specific reduced equations these algebras are not, generally speaking, the maximal ones. We will consider in more detail symmetry properties of the systems 5–7 from (2.3.3).

By the Lie method we can prove the following assertions.

Theorem 2.3.2. *Equation 5 from (2.3.3) is invariant under the conformal group $C(3)$ iff*

$$
f_j = (\bar\psi\psi)^{1/2}\tilde f_j\Big(\bar\psi\psi(\bar\psi\gamma_4\psi)^{-1}\Big), \quad j = 1, 2. \qquad (2.3.14)
$$

Theorem 2.3.3. *Equation 6 from (2.3.3) is invariant under the conformal group $C(1,2)$ iff (2.3.14) holds.*

Theorem 2.3.4. *Equation 7 from (2.3.3) admits an infinite-parameter invariance group with the following generators:*

a) *with arbitrary* $f_1,\ f_2$

$$\begin{aligned}&Q_1 = \partial_{\omega_1}, \quad Q_2 = -\omega_2\partial_{\omega_3} + \omega_3\partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\\ &Q_3 = w_1\partial_{\omega_2} + w_2\partial_{\omega_3} + (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0 + \gamma_3),\\ &Q_4 = \omega_1\partial_{\omega_1} - (1/2)\gamma_0\gamma_3;\end{aligned} \tag{2.3.15}$$

b) *with* $f_1 = f_1(\bar{\psi}\psi),\ f_2 = 0$

$$\begin{aligned}&Q_1 = \partial_{\omega_1}, \quad Q_2 = -\omega_2\partial_{\omega_3} + \omega_3\partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\\ &Q_3 = w_1\partial_{\omega_2} + w_2\partial_{\omega_3} + (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0 + \gamma_3),\\ &Q_4 = w_3\gamma_4(\gamma_0 + \gamma_3), \quad Q_5 = \omega_1\partial_{\omega_1} - (1/2)\gamma_0\gamma_3;\end{aligned}$$

c) *with* $f_i = (\bar{\psi}\psi)^{1/2k}\tilde{f}_i\Big(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}\Big),\ i = 1, 2$

$$\begin{aligned}&Q_1 = \partial_{\omega_1}, \quad Q_2 = -\omega_2\partial_{\omega_3} + \omega_3\partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\\ &Q_3 = w_1\partial_{\omega_2} + w_2\partial_{\omega_3} + (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0 + \gamma_3),\\ &Q_4 = \omega_a\partial_{\omega_a} + k, \quad Q_5 = \omega_1\partial_{\omega_1} - (1/2)\gamma_0\gamma_3;\end{aligned}$$

d) *with* $f_i = (\bar{\psi}\psi)^{1/2}\tilde{f}_i\Big(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}\Big),\ i = 1, 2$

$$\begin{aligned}&Q_1 = w_1\partial_{\omega_2} + w_2\partial_{\omega_3} + (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0 + \gamma_3),\\ &Q_2 = -\omega_2\partial_{\omega_3} + \omega_3\partial_{\omega_2} + (1/2)\gamma_1\gamma_2,\\ &Q_3 = w_0\partial_{\omega_1} + \dot{w}_0(\omega_2\partial_{\omega_2} + \omega_3\partial_{\omega_3}) + \dot{w}_0 + (1/2)\ddot{w}_0\\ &\quad\times(\gamma_1\omega_2 + \gamma_2\omega_3)(\gamma_0 + \gamma_3), \quad Q_4 = \omega_1\partial_{\omega_1} - (1/2)\gamma_0\gamma_3.\end{aligned}$$

Here $w_\mu = w_\mu(\omega_1)$ are arbitrary smooth real-valued functions, an overdot means differentiation with respect to ω_1.

Consequently, the invariance algebras of PDEs 5–7 from (2.3.3) are substantially wider than the algebras $\widetilde{A}_5 - \widetilde{A}_7$ adduced in (2.3.12).

Using the above results we have constructed the Ansätze for field $\psi(x)$ reducing PDE (2.3.1) with $f_1 = f_1(\bar{\psi}\psi)$, $f_2 = 0$ which cannot be obtained within the framework of the Lie approach:

1) $f_1 \in C(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function

$$\begin{aligned}\psi(x) \;=\; &\exp\{w_3\gamma_4(\gamma_0 + \gamma_3) - (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0 + \gamma_3)\}\\ &\times\begin{cases}\varphi(x_1 + w_1),\\ \exp\{-(1/2)\gamma_1\gamma_2\arctan[(x_1 + w_1)(x_2 + w_2)^{-1}]\}\\ \quad\times\varphi\,[(x_1 + w_1)^2 + (x_2 + w_2)^2];\end{cases}\end{aligned}$$

2) $f_1 = \lambda(\bar{\psi}\psi)^{1/2}, \ \lambda \in \mathbb{R}^1$

$$
\begin{aligned}
\psi(x) \ &= \ w_0^{-1} \exp\{w_3\gamma_4(\gamma_0+\gamma_3) - (1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2)(\gamma_0+\gamma_3) \\
&\quad -(\dot{w}_0/2w_0)[\gamma_1(x_1+w_1)+\gamma_2(x_2+w_2)](\gamma_0+\gamma_3)\} \qquad (2.3.16)\\
&\quad \times \begin{cases} \varphi\,[w_0^{-1}(x_1+w_1)], \\ \exp\{-(1/2)\arctan[(x_1+w_1)(x_2+w_2)^{-1}]\} \\ \quad \times\varphi\,[(x_1+w_1)^2 w_0^{-2} + (x_2+w_2)^2 w_0^{-2}]; \end{cases}\\
\psi(x) \ &= \ (\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2)(x_0^2 - x_1^2 - x_2^2)^{-3/2}\\
&\quad \times \begin{cases} \varphi\,[x_0(x_0^2-x_1^2-x_2^2)^{-1}], \\ \varphi\,[x_1(x_0^2-x_1^2-x_2^2)^{-1}], \\ \exp\{-(1/2)\arctan(x_1/x_2)\} \\ \quad \times\varphi\,[(x_1^2+x_2^2)(x_0^2-x_1^2-x_2^2)^{-2}]; \end{cases}\\
\psi(x) \ &= \ (\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3)(x_1^2 + x_2^2 + x_3^2)^{-3/2}\\
&\quad \times \begin{cases} \varphi\,[x_1(x_1^2+x_2^2+x_3^2)^{-1}], \\ \exp\{-(1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\} \\ \quad \times\varphi\,[(x_1^2+x_2^2)(x_1^2+x_2^2+x_3^2)^{-2}]. \end{cases}
\end{aligned}
$$

In (2.3.16) $w_0, \ldots, w_3$ are arbitrary smooth functions of x_0+x_3; $\varphi = \varphi\,(\omega)$ are unknown four-component functions.

Substitution of Ansätze (2.3.16) into PDE (2.3.1) with corresponding $f_1, \ f_2$ gives rise to the following systems of ODEs:

$$
\begin{aligned}
&i\gamma_1\dot{\varphi} = f_1(\bar{\varphi}\varphi)\varphi,\\
&(i/2)\omega^{-1/2}\gamma_2\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = f_1(\bar{\varphi}\varphi)\varphi,\\
&i\gamma_1\dot{\varphi} = \lambda(\bar{\varphi}\varphi)^{1/2}\varphi,\\
&(i/2)\omega^{-1/2}\gamma_2\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = \lambda(\bar{\varphi}\varphi)^{1/2}\varphi,\\
&i\gamma_0\dot{\varphi} = -\lambda(\bar{\varphi}\varphi)^{1/2}\varphi, \qquad (2.3.17)\\
&i\gamma_1\dot{\varphi} = -\lambda(\bar{\varphi}\varphi)^{1/2}\varphi,\\
&(i/2)\omega^{-1/2}\gamma_2\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = -\lambda(\bar{\varphi}\varphi)^{1/2}\varphi,\\
&i\gamma_1\dot{\varphi} = -\lambda(\bar{\varphi}\varphi)^{1/2}\varphi,\\
&(i/2)\omega^{-1/2}\gamma_2\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = -\lambda(\bar{\varphi}\varphi)^{1/2}\varphi.
\end{aligned}
$$

From Theorem 2.3.4 it follows that the Dirac equation (1.1.1) is conditionally-invariant under an infinite-parameter Lie group. As established in [152, 155] a broad class of Poincaré-invariant equations (the Bhabha-type equations)

$$(i\beta_s\partial_s - m)\Psi(x) = 0, \quad m = \text{const} \qquad (2.3.18)$$

possess such a property.

In (2.3.18) $\Psi = (\Psi^1, \Psi^2, \dots, \Psi^n)^T$; $x = (x_0, x_1, \dots, x_N)$, $N \geq 2$; $\beta_0,\ \beta_1, \dots, \beta_N$ are $(n \times n)$-matrices satisfying the conditions

$$[\beta_s,\ S_{\tau\rho}] = (g_{s\tau}\beta_\rho - g_{s\rho}\beta_\tau), \tag{2.3.19}$$

where $S_{\tau\rho} = (\beta_\tau\beta_\rho - \beta_\rho\beta_\tau),\ g_{s\tau} = \mathrm{diag}\,(1, -1, -1, \dots, -1)$.

It is well-known that the Bhabha equation is invariant under the Poincaré group $P(1,N)$ having the generators [30]

$$P_\tau = g_{\tau\rho}\partial_{x_\rho}, \quad J_{\tau\rho} = x_\tau P_\rho - x_\rho P_\tau + S_{\tau\rho}.$$

Imposing an additional condition $(\partial_{x_0} - \partial_{x_N})\Psi(x) = 0$ on $\Psi(x)$ we get the following system of PDEs for $\Psi(\omega) = \Psi(x_0 + x_N,\ x_1, \dots, x_{N-1})$:

$$\left\{ i(\beta_0 + \beta_N)\partial_{\omega_0} + \sum_{j=1}^{N-1} \beta_j \partial_{\omega_j} - m \right\} \Psi(\omega) = 0. \tag{2.3.20}$$

Theorem 2.3.5. *Equation (2.3.20) is invariant under the infinite-parameter Lie group having the generators*

$$\begin{aligned} &Q_1 = \partial_{\omega_0}, \quad Q_{jk} = -\omega_j\partial_{\omega_k} + \omega_k\partial_{\omega_j} + S_{jk}, \\ &Q_2 = \sum_{k=1}^{N-1} \{W_k(\omega_0)\partial_{\omega_k} - \dot{W}_k(\omega_0)(S_{0k} - S_{kN})\}, \end{aligned} \tag{2.3.21}$$

where $W_1,\ W_2, \dots, W_{N-1}$ are arbitrary smooth functions, $\dot{W}_k = dW_k/d\omega_0$, j, $k = 1, \dots, N-1$.

Proof. It is evident that the operators $Q_1,\ Q_{jk}$ belong to the invariance algebra of equation (2.3.20). Let us prove that the operator Q_2 commutes with the operator of equation (2.3.21)

$$L = i(\beta_0 + \beta_N)\partial_{\omega_0} + i\sum_{j=1}^{N-1} \beta_j \partial_{\omega_j} - m.$$

Computing the commutator $[L, Q]$ we have

$$\begin{aligned} [L, Q] \ &= \ i\sum_{k=1}^{N-1} \Big\{ -\ddot{W}_k(\beta_0 + \beta_N)(S_{0k} - S_{kN}) \\ &\quad -\dot{W}_k[\beta_0 + \beta_N,\ S_{0k} - S_{kN}]\partial_{\omega_0} \Big\}. \end{aligned}$$

Resulting from relations (2.3.19), the equalities

$$\begin{aligned}
&[\beta_0+\beta_N,\, S_{0k}-S_{kN}]=0,\\
&(\beta_0+\beta_N)(S_{0k}-S_{kN})=(\beta_0+\beta_N)(\beta_0\beta_k-\beta_k\beta_0-\beta_k\beta_N\\
&\quad+\beta_N\beta_k)=(\beta_0\beta_0\beta_k-\beta_0\beta_k\beta_0)+(\beta_N\beta_N\beta_k-\beta_N\beta_k\beta_N)\\
&\quad+(\beta_0\beta_N\beta_k-\beta_N\beta_k\beta_0)+(\beta_N\beta_0\beta_k-\beta_0\beta_k\beta_N)\\
&\quad=\beta_k-\beta_k=0,\quad k=1,\ldots,N-1
\end{aligned}$$

hold, whence it follows that $[L,Q]=0$. The theorem is proved. $\triangleright$

$\widetilde{P}(1,3)$-invariant Ansätze for the spinor field $\psi=\psi(x)$ (2.2.8) obtained in the previous section reduce a $\widetilde{P}(1,3)$-invariant spinor equation

$$i\gamma_\mu\partial_\mu\psi-(\bar\psi\psi)^{1/2k}\Big\{\tilde f_1\Big(\bar\psi\psi(\bar\psi\gamma_4\psi)^{-1}\Big)+\tilde f_2\Big(\bar\psi\psi(\bar\psi\gamma_4\psi)^{-1}\Big)\gamma_4\Big\}\psi=0$$

to systems of ODEs of the form

$$\begin{aligned}
&1)\quad 2i\gamma_3\dot\varphi+(i/4)(\gamma_0+\gamma_3)(\gamma_0\gamma_3-2k)\varphi=R,\\
&2)\quad i(\gamma_0-2\gamma_2-\gamma_3)\dot\varphi+(i/2)\gamma_2(\gamma_0\gamma_3-2k)\varphi=R,\\
&3)\quad 2i\gamma_3\dot\varphi+(i/4\alpha)(\gamma_0+\gamma_3)(\alpha\gamma_0\gamma_3-\gamma_1\gamma_2-2k\alpha)\varphi=R,\\
&4)\quad (i/2)(\gamma_0-\gamma_3-2\gamma_1+2\alpha\gamma_2)\dot\varphi+(i/2)(1-2k+\gamma_0\gamma_3)\varphi=R,\\
&5)\quad (i/2)(\gamma_0-\gamma_3+2\gamma_2)\dot\varphi+(i/2)\gamma_1(1-2k+\gamma_0\gamma_3)\varphi=R,\\
&6)\quad i\omega(4\omega\gamma_1+\gamma_2)\dot\varphi+(1/4)\gamma_2(\gamma_0\gamma_3-4k)\varphi=R,\\
&7)\quad -i\omega\Big(12\gamma_1+\omega^{1/2}(15\gamma_0+9\gamma_3)\Big)\dot\varphi-i\gamma_1(\gamma_0\gamma_3-4k)\varphi=R,\\
&8)\quad i(\gamma_0-\gamma_3)\dot\varphi+(i/2\omega)\Big(\gamma_0-\gamma_3+(\gamma_1-\omega\gamma_2)\\
&\qquad\times(\gamma_0\gamma_3-2k)\Big)\varphi=R,\\
&9)\quad 2i\gamma_0\dot\varphi+(i/4)(\gamma_0-\gamma_3)(2-2k-\gamma_0\gamma_3)\varphi=R,\\
&10)\quad i(\gamma_0-\gamma_3)\dot\varphi+(i/2)(\gamma_0-\gamma_3)\Big(\omega^{-1}+(\omega+1)^{-1}\Big)\varphi+(i/4)\\
&\qquad\times\Big((\gamma_0+\gamma_3)(1+\omega)+(\gamma_0-\gamma_3)(1+\omega)^{-1}\Big)(\gamma_0\gamma_3-2k)\varphi=R,\\
&11)\quad 2i\gamma_0\dot\varphi+(i/4)(\gamma_0-\gamma_3)(4-2k-\gamma_0\gamma_3)\varphi=R,\\
&12)\quad 2i\gamma_0\dot\varphi+(i/4\alpha)(\gamma_0-\gamma_3)\Big((4-2k)\alpha-\alpha\gamma_0\gamma_3+\gamma_1\gamma_2\Big)\varphi=R,\\
&13)\quad i(\alpha\gamma_2-\gamma_1)\dot\varphi+(i/2)(1-2k)\gamma_1\varphi=R,\\
&14)\quad i(\gamma_0-\gamma_3\omega)\dot\varphi+(i/2\alpha)\gamma_3(\gamma_1\gamma_2-2k\alpha)\varphi=R,\\
&15)\quad i(\gamma_1-\gamma_2\omega)\dot\varphi-(i/2\alpha)\gamma_2(2k\alpha+\gamma_0\gamma_3)\varphi=R,
\end{aligned}$$

$$
\begin{aligned}
&16)\quad (i/\alpha)\Big(\alpha(\gamma_0-\gamma_3)-(\alpha+1)\omega\gamma_2\Big)\dot\varphi-(i/2\alpha)\gamma_2(2k\alpha+\gamma_0\gamma_3)\varphi=R,\\
&17)\quad i\omega^2(\gamma_0+\alpha\gamma_3)\dot\varphi+(i/2\alpha)\omega(\gamma_0+2k\alpha\gamma_3)\varphi=R,\\
&18)\quad i(\beta\gamma_2-\gamma_1)\dot\varphi+(i/2\beta)\gamma_1\Big((1-2k)\beta-\alpha\gamma_0\gamma_3\Big)\varphi=R,\\
&19)\quad i\omega^{(\alpha+1)/\alpha}(\alpha\gamma_0+\beta\gamma_3)\dot\varphi+(i/2\alpha)\omega^{1/\alpha}\gamma_3(\alpha\gamma_0\gamma_3-\gamma_1\gamma_2\\
&\qquad +2\beta k)\varphi=R,\\
&20)\quad i(\gamma_0-\gamma_3+\alpha\gamma_1)\dot\varphi+(i/2)(1-2k)(\gamma_0-\gamma_3)\varphi=R,\\
&21)\quad i(\gamma_0-\gamma_3-\omega\gamma_2)\dot\varphi-ik\gamma_2\varphi=R,\\
&22)\quad (i/\alpha)\Big(\alpha(\gamma_2-\beta\gamma_1)+\beta(\gamma_3-\gamma_0)\Big)\dot\varphi-(i/2\beta)(\gamma_0-\gamma_3)(2k\beta+\gamma_1\gamma_2)\varphi\\
&\qquad =R,\\
&23)\quad i(\alpha\gamma_1+\beta\gamma_2+\gamma_0-\gamma_3)\dot\varphi+(i/\omega)(\gamma_0-\gamma_3)\varphi=R,\\
&24)\quad (i/\alpha)\Big(\alpha(\gamma_0-\gamma_3)-(\alpha+1)\omega\gamma_2\Big)\dot\varphi+(i/2\omega)(\gamma_0-\gamma_3)\varphi\\
&\qquad -(i/2\alpha)\gamma_2(\gamma_0\gamma_3+2k\alpha)\varphi=R,\\
&25)\quad (i/2\alpha)\Big((\alpha-1)(\gamma_0-\gamma_3)-(\alpha+1)\omega^2(\gamma_0+\gamma_3)\Big)\dot\varphi+(i/2\omega)(\gamma_0-\gamma_3)\varphi\\
&\qquad -(i/4\alpha\omega)\Big((\gamma_0-\gamma_3)+(\gamma_0+\gamma_3)\omega^2\Big)(\gamma_0\gamma_3+2k\alpha)\varphi=R,\\
&26)\quad i\Big((\beta+1)\gamma_1-\beta(\gamma_0-\gamma_3)-\alpha\gamma_2\Big)\dot\varphi+(i/2)\gamma_1(1-2k+\gamma_0\gamma_3)\varphi\\
&\qquad +(i/2)(\gamma_0-\gamma_3)\varphi=R, \qquad\qquad (2.3.22)\\
&27)\quad (i/2)(1-\omega)^{-1}(\omega^{-1/2}\gamma_0-\gamma_2)\dot\varphi+i\Big((1/2)(\omega^{-1/2}\gamma_0+\gamma_2)\\
&\qquad +k(\omega-1)^{-1}(\omega^{1/2}\gamma_0-\gamma_1)\Big)\varphi=R(\omega-1)^{-1/2},\\
&28)\quad i(\gamma_1-\omega\gamma_2)\dot\varphi-ik(\omega^2+1)^{-1}(\omega\gamma_1+\gamma_2)\varphi=R(\omega^2+1)^{-1/2},\\
&29)\quad (i\gamma_0-\omega\gamma_3)\dot\varphi-ik(\omega^2-1)^{-1}(\omega\gamma_0-\gamma_3)=R(\omega^2-1)^{-1/2},\\
&30)\quad i\Big(\gamma_2-\omega(\gamma_0-\gamma_3)\Big)\dot\varphi-ik(\gamma_0-\gamma_3)\varphi=R,\\
&31)\quad i(\omega\gamma_1-\omega^2\gamma_3)\dot\varphi+(i/2)(1-2k)\gamma_1\varphi=R,\\
&32)\quad i(\omega\gamma_1-\omega^2\gamma_0)\dot\varphi+(i/2)(1-2k)\gamma_1\varphi=R,\\
&33)\quad i\Big(\omega\gamma_1-\omega^2(\gamma_0-\gamma_3)\Big)\dot\varphi+(i/2)(1-2k)\gamma_1\varphi=R,\\
&34)\quad i(\omega\gamma_0-\omega^2\gamma_2\Big)\dot\varphi+(i/2)(1-2k)\gamma_1\varphi=R,\\
&35)\quad i(\gamma_2-\omega\gamma_1)\dot\varphi+(i/2)(\gamma_0-\gamma_3+\gamma_2\gamma_4-2k\gamma_1)\varphi=R,\\
&36)\quad i(\gamma_1+\alpha\gamma_2-\gamma_0+\gamma_3)\dot\varphi+(i/2)\Big(\gamma_0-\gamma_3+\gamma_2\gamma_4+(1-2k)\gamma_1\Big)\varphi=R,\\
&37)\quad i\Big(\gamma_2-\omega(\gamma_0-\gamma_3)\Big)\dot\varphi+(i/2)(1-2k)(\gamma_0-\gamma_3)\varphi=R,
\end{aligned}
$$

38) $i\Big(\gamma_0+\gamma_3-\omega(\gamma_0-\gamma_3)\Big)\dot\varphi+(i/2)(1-2k)(\gamma_0-\gamma_3)\varphi=R,$

39) $i\Big(\gamma_0+\gamma_3-\omega(\gamma_0-\gamma_3)\Big)\dot\varphi+i(1-k)(\gamma_0-\gamma_3)\varphi=R,$

40) $i\Big(\omega(\gamma_0-2\gamma_2-\gamma_3)+\gamma_0+\gamma_3\Big)\dot\varphi+(i/2)\Big(2(\gamma_0-\gamma_3)-\gamma_1\gamma_4-2k\gamma_2\Big)\varphi$
$=R,$

41) $i\Big((1-\omega)(\gamma_0-\gamma_3)+\gamma_2\Big)\dot\varphi+(i/2)(1-2k)(\gamma_0-\gamma_3)\varphi=R,$

42) $i\omega\Big((1-\alpha)\omega^{1/2\alpha}(\gamma_0-\gamma_3)+(1+\alpha)\omega^{-1/2\alpha}(\gamma_0+\gamma_3)\Big)\dot\varphi$
$+(i/4\alpha)\Big[\Big(1+2\alpha(2-k)\Big)\omega^{1/2\alpha}(\gamma_0-\gamma_3)-(1+2k\alpha)\omega^{-1/2\alpha}$
$\times(\gamma_0+\gamma_3)\Big]\varphi=R,$

43) $i\Big(\gamma_0+\gamma_3-\omega(\gamma_0-\gamma_3)\Big)\dot\varphi+(i/2\alpha)\Big(2\alpha(1-k)+\gamma_4\Big)(\gamma_0-\gamma_3)\varphi=R,$

44) $i\omega\Big((\alpha-\beta)\omega^{1/2\beta}(\gamma_0-\gamma_3)+(\alpha+\beta)\omega^{-1/2\beta}(\gamma_0+\gamma_3)\Big)\dot\varphi$
$+(i/4\beta)\Big[\Big(\alpha+4\beta(1-k)-\gamma_4\Big)\omega^{1/2\beta}(\gamma_0-\gamma_3)$
$-(\alpha+4k\beta-\gamma_4)\omega^{-1/2\beta}(\gamma_0+\gamma_3)\Big]\varphi=R,$

where $R=(\bar\varphi\varphi)^{1/2k}\{\tilde f_1(\bar\varphi\varphi(\bar\varphi\gamma_4\varphi)^{-1})+\tilde f_2(\bar\varphi\varphi(\bar\varphi\gamma_4\varphi)^{-1})\gamma_4\}\varphi.$

At last, Ansätze (2.2.29) invariant under $C(1,3)$ non-conjugate three-dimensional subalgebras of the algebra $AC(1,3)$ listed in (2.2.29) after being substituted into a conformally-invariant spinor equation

$$i\gamma_\mu\partial_\mu\psi-(\bar\psi\psi)^{1/3}\Big\{\tilde f_1\Big(\bar\psi\psi(\bar\psi\gamma_4\psi)^{-1}\Big)+\tilde f_2\Big(\bar\psi\psi(\bar\psi\gamma_4\psi)^{-1}\Big)\gamma_4\Big\}\psi=0$$

give rise to the following systems of ODEs for $\varphi=\varphi(\omega)$:

$$\begin{aligned}
&1)\quad i\Big(-(3/4)(\omega^2+4)(\gamma_0-\gamma_3)+\gamma_0+\gamma_3+\omega\gamma_1+2\gamma_2\Big)\dot\varphi\\
&\qquad +i\Big(\gamma_1-\omega(\gamma_0-\gamma_3)+(1/2)\gamma_1\gamma_2(\gamma_0-\gamma_3)\Big)\varphi=R,\\
&2)\quad i\Big(\gamma_1\cos\omega-\gamma_2\sin\omega-\alpha(\gamma_0-\gamma_3)\Big)\dot\varphi-(i/2)(3-\gamma_0\gamma_3)\\
&\qquad \times(\gamma_1\sin\omega+\gamma_2\cos\omega)\varphi-(i\alpha/2)(\gamma_0-\gamma_3)\gamma_1\gamma_2\varphi=R,\\
&3)\quad i(\gamma_2-\gamma_0+\gamma_3)\dot\varphi=R,\\
&4)\quad i\Big(\gamma_1+\alpha(\gamma_0-\gamma_3)e^{-\omega}\Big)\dot\varphi+2i\gamma_1\varphi=Re^{\omega/3},\\
&5)\quad i\Big(\gamma_1+\alpha\gamma_2-\beta(\gamma_0-\gamma_3)e^{\omega}\Big)\dot\varphi-(i/2)\Big(3\gamma_1\\
&\qquad +2\alpha\gamma_2(1+\gamma_0\gamma_3)\Big)\varphi=Re^{-\omega/3},
\end{aligned}\tag{2.3.23}$$

6) $i(\omega^2\gamma_2 - \omega\gamma_1)\dot{\varphi} - i\omega(1+\omega^2)^{-1}\gamma_2(1+\gamma_0\gamma_3)\varphi + (i/2)(\omega^2+3)$
$\times(\omega^2+1)^{-1}\gamma_1\varphi - (i/2)(3\omega^2+1)(\omega^2+1)^{-1}\gamma_2\gamma_4\varphi = R,$

7) $i\Big((\omega^2+1)^{1/4}(\gamma_0+\gamma_3) + (\omega^2+1)^{-1/4}(\gamma_0-\gamma_3) - 2\omega\gamma_2\Big)\dot{\varphi}$
$+i\Big(\omega(\omega^2+1)^{-3/4}(\gamma_0+\gamma_3) - (\omega^2+1)^{-1}(1+\gamma_0\gamma_3)\gamma_2$
$-(3/2)\gamma_2\Big)\varphi = R(\omega^2+1)^{1/4},$

8) $2i\omega\gamma_2\dot{\varphi} - (3i/2)\gamma_2\varphi = R\omega^{-1/6},$

9) $-i\gamma_1\dot{\varphi} = R,$

10) $-2i\omega(\omega-4)\gamma_2\dot{\varphi} - i\Big((1/2)\omega^{1/2}(\omega-4)^{1/2} + \omega\ -2\Big)\gamma_2\varphi$
$= R\omega^{1/2}(\omega-4)^{1/2}\Big((1/2)\Big[\omega^{1/2} + (\omega-4)^{1/2}\Big]\Big)^{1/3},$

where $R = (\bar{\varphi}\varphi)^{1/3}\Big\{\tilde{f}_1\Big(\bar{\varphi}\varphi(\bar{\varphi}\gamma_4\varphi)^{-1}\Big) + \tilde{f}_2\Big(\bar{\varphi}\varphi(\overline{\gamma}_4\varphi)^{-1}\Big)\gamma_4\Big\}\varphi.$

2.4. Exact solutions of nonlinear spinor equations

Using the results obtained in Sections 2.2, 2.3 we will construct in explicit form multi-parameter families of exact solutions of the following systems of nonlinear PDEs:

$$\{i\gamma_\mu\partial_\mu - \lambda(\bar{\psi}\psi)^{1/2k}\}\psi = 0, \tag{2.4.1}$$

$$\{i\gamma_\mu\partial_\mu - m - \lambda(\bar{\psi}\psi)^{k}\}\psi = 0, \tag{2.4.2}$$

which are obtained from (2.3.1) by putting $f_1 = \lambda(\bar{\psi}\psi)^{1/2k}$, $f_2 = 0$ and $f_1 = m + \lambda(\bar{\psi}\psi)^k$, $f_2 = 0$, respectively.

In (2.4.1), (2.4.2) $m,\ \lambda,\ k$ are real constants, $m \neq 0$, $k \neq 0$.

Equation (2.4.1) with $k = 1/2$ was considered by Heisenberg [180]–[74] (see also [184]) and equation (2.4.2) with $k = 1$ was suggested by Ivanenko as a possible basic model for the unified field theory [192].

According to Theorem 1.2.1 equations (2.4.1), (2.4.2) are invariant under the Poincaré group. In addition, system of PDEs (2.4.1) admits the one-parameter group of scale transformations (1.1.28).

To reduce equations (2.4.1), (2.4.2) we apply $P(1,3)$-, $\widetilde{P}(1,3)$- and $C(1,3)$-invariant Ansätze constructed in Section 2.2.

1. Poincaré-invariant solutions of system of PDEs (2.4.1).

1.1. *Integration of reduced ODEs.* Substitution of the $P(1,3)$-invariant Ansätze (2.2.8) into (2.4.1) gives rise to systems of ODEs (2.3.5) with $R = -i\lambda(\bar{\varphi}\varphi)^{1/2k} \times\varphi$. When integrating these we will use essentially the following assertions.

Lemma 2.4.1. *Solutions of equations 3, 12–14, 20–23 from (2.3.5) satisfy the relation* $\bar{\varphi}\varphi = 0$.

Proof. Multiplication of the ODE 3 from (2.3.5) by the matrix $\gamma_0 + \gamma_3$ on the left yields the following consistency condition:

$$-i\lambda(\bar{\varphi}\varphi)^{1/2k}(\gamma_0 + \gamma_3)\varphi = 0,$$

whence $\bar{\varphi}\varphi = 0$ or $(\gamma_0 + \gamma_3)\varphi = 0$. The general solution of the algebraic equation $(\gamma_0 + \gamma_3)\varphi = 0$ is represented in the form

$$\varphi = (\gamma_0 + \gamma_3)\varphi_1,$$

where φ_1 is an arbitrary four-component function-column.

Since $\bar{\varphi} = \{\varphi_1(\gamma_0 + \gamma_3)\}^\dagger\gamma_0 = \bar{\varphi}_1(\gamma_0 + \gamma_4)$, an identity $\bar{\varphi}\varphi = \bar{\varphi}_1(\gamma_0 + \gamma_4)^2\varphi_1 = 0$ holds. Other equations are treated in the same way. $\triangleright$

Lemma 2.4.2. *The quantity* $\bar{\varphi}\varphi$ *is the first integral of systems of ODEs 1, 2, 5, 15, 16, 25 from (2.3.5).*

We prove the assertion for the system 1. Multiplying it by $-\gamma_3$ yields

$$\dot{\varphi} = i\lambda(\bar{\varphi}\varphi)^{1/2k}\gamma_3\varphi. \tag{2.4.3}$$

The conjugate spinor satisfies the following equation:

$$\dot{\bar{\varphi}} = -i\lambda(\bar{\varphi}\varphi)^{1/2k}\bar{\varphi}\gamma_3. \tag{2.4.4}$$

Multiplying (2.4.3) by $\bar{\varphi}$ on the left, (2.4.4) by φ on the right and summing the expressions obtained we arrive at the relation

$$\dot{\bar{\varphi}}\varphi + \bar{\varphi}\dot{\varphi} = 0,$$

whence $d(\bar{\varphi}\varphi)/d\omega = \dot{\bar{\varphi}}\varphi + \bar{\varphi}\dot{\varphi} = 0$. The lemma is proved. $\triangleright$

Due to Lemma 2.4.1 we conclude that the Ansätze numbered by 3, 12–14, 20–23 give rise to the solutions of equation (2.4.1) which satisfy the condition $\bar{\psi}\psi = \bar{\varphi}\varphi = 0$. Consequently, a factor $\lambda(\bar{\psi}\psi)^{1/2k}$ determining the nonlinear self-coupling of the spinor field $\psi(x)$ vanishes. Such solutions are of low interest and are not considered here.

According to Lemma 2.4.2 the system of ODEs 1 from (2.3.5) is equivalent to the linear equation

$$\dot{\varphi} = i\lambda C^{1/2k}\gamma_3\varphi \tag{2.4.5}$$

with a nonlinear additional constraint

$$\bar{\varphi}\varphi = C = \text{const.} \tag{2.4.6}$$

Integrating ODE (2.4.5) we get

$$\varphi(\omega) = \exp\{i\lambda C^{1/2k}\gamma_3\omega\}\chi, \quad \bar{\varphi}(\omega) = \bar{\chi}\,\exp\{-i\lambda C^{1/2k}\gamma_3\omega\}. \tag{2.4.7}$$

Hereafter χ is an arbitrary constant four-component column.

Substitution of expressions (2.4.7) into (2.4.6) yields

$$\bar{\chi}\exp\{-i\lambda C^{1/2k}\gamma_3\omega\}\exp\{i\lambda C^{1/2k}\gamma_3\omega\}\chi = C,$$

whence $\bar{\chi}\chi = C$. Thus, the general solution of the system of nonlinear ODEs 1 from (2.3.5) is given by the formula

$$\varphi(\omega) = \exp\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_3\omega\}\chi.$$

The general solutions of equations 2, 5, 15, 16, 25 are constructed in the same way. As a result, we have

$$\begin{aligned}
\varphi(\omega) &= \exp\{-i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_0\omega\}\chi,\\
\varphi(\omega) &= \exp\{i\gamma_2\Big((\bar{\chi}\chi)^{1/2k} - (i/2)(\gamma_0+\gamma_3)\Big)\omega\}\chi,\\
\varphi(\omega) &= \exp\{(i\lambda/2)(\bar{\chi}\chi)^{1/2k}\gamma_1\omega\}\chi,\\
\varphi(\omega) &= \exp\{(i\lambda/2)(1+\alpha^2)^{-1}(\bar{\chi}\chi)^{1/2k}(\gamma_2-\alpha\gamma_1)\omega\}\chi,\\
\varphi(\omega) &= \exp\Big\{\Big[\gamma_2(\gamma_0+\gamma_3) + i\lambda(\bar{\chi}\chi)^{1/2k}\Big(\gamma_2-\beta(\gamma_0+\gamma_3)\Big)\Big]\omega\Big\}\chi.
\end{aligned} \tag{2.4.8}$$

To integrate systems of ODEs 6, 9–11 from (2.3.5) we will use their symmetry properties. As established in Section 2.3 the equation 6 is invariant under the Lie algebra with the basis elements ∂_ω, $\gamma_0\gamma_3$. We seek for a solution which is invariant under the one-dimensional subalgebra of this algebra $\langle\partial_\omega - \theta\gamma_0\gamma_3\rangle$, $\theta \in \mathbb{R}^1$.

In other words, a four-component function $\varphi = \varphi(\omega)$ has to satisfy the additional constraint

$$Q\varphi = (\partial_\omega - \theta\gamma_0\gamma_3)\varphi = 0.$$

The general solution of the above equation reads

$$\varphi(\omega) = \exp\{\theta\gamma_0\gamma_3\omega\}\chi', \qquad (2.4.9)$$

where χ' is an arbitrary constant four-component column. Substituting (2.4.9) into the system of ODEs 6 from (2.3.5) we have

$$\Big(\theta\gamma_1\gamma_0\gamma_3 - (1/2\alpha)\gamma_1\gamma_4\Big)\exp\{\theta\gamma_0\gamma_3\omega\}\chi' = -i\lambda\tau\exp\{\theta\gamma_0\gamma_3\omega\}\chi',$$

where $\tau = (\bar{\chi}'\chi')^{1/2k}$.

Multiplying both parts of the above equality by $\exp\{-\theta\gamma_0\gamma_3\omega\}$ on the left we arrive at the system of algebraic equations for χ'

$$\Big\{\Big(\theta\gamma_2 - (1/2\alpha)\gamma_1\Big)\gamma_4 + i\lambda\tau\Big\}\chi' = 0. \qquad (2.4.10)$$

Consequently, substitution (2.4.9) reduces the system 6 to algebraic equations (2.4.10). Making in (2.4.10) the transformation

$$\chi' = \Big([\theta\gamma_2 - (1/2\alpha)\gamma_1]\gamma_4 - i\lambda\tau\Big)\chi$$

yields

$$[\lambda^2\tau^2 - \theta^2 - (2\alpha)^{-2}]\chi = 0.$$

As $\chi \neq 0$, the equality

$$\theta = (\varepsilon/2\alpha)(4\lambda^2\tau^2\alpha^2 - 1)^{1/2}, \quad \varepsilon = \pm 1 \qquad (2.4.11)$$

has to be satisfied.

The condition $\tau = (\bar{\chi}'\chi')^{1/2k}$ gives rise to the nonlinear algebraic equation for τ

$$\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_2\gamma_4\chi) - i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_1\gamma_4\chi). \qquad (2.4.12)$$

Thus, we have constructed a particular solution of the system of ODEs 6 from (2.3.5)

$$\varphi(\omega) = \exp\{\theta\gamma_0\gamma_3\omega\}\Big([\theta\gamma_2 - (1/2\alpha)\gamma_1]\gamma_4 - i\lambda\tau\Big)\chi,$$

where $\theta,\ \tau$ are determined by (2.4.11), (2.4.12).

Particular solutions of systems of ODEs 9–11 from (2.3.5) are obtained in an analogous way

$$\varphi(\omega) = \exp\{\theta\gamma_1\gamma_2\omega\}\Big([\theta\gamma_0 - (1/2\alpha)\gamma_3]\gamma_4 - i\lambda\tau\Big)\chi, \qquad (2.4.13)$$

parameters $\theta,\ \tau$ being defined by the formulae

$$\begin{aligned}&\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_0\gamma_4\chi) - i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_3\gamma_4\chi),\\ &\theta = (\varepsilon/2\alpha)(1 - 4\alpha^2\lambda^2\tau^2)^{1/2};\end{aligned} \tag{2.4.14}$$

$$\varphi(\omega) = \exp\{\theta\gamma_1\gamma_2\omega\}\Big([\theta\gamma_3 + (1/2\alpha)\gamma_0]\gamma_4 - i\lambda\tau\Big)\chi, \tag{2.4.15}$$

parameters $\theta,\ \tau$ being defined by the formulae

$$\begin{aligned}&\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_3\gamma_4\chi) + i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_0\gamma_4\chi),\\ &\theta = (\varepsilon/2\alpha)(1 + 4\alpha^2\lambda^2\theta^2)^{1/2};\end{aligned} \tag{2.4.16}$$

$$\varphi(\omega) = \exp\{\theta\gamma_1\gamma_2\omega\}\Big(4\theta(\gamma_0 + \gamma_3)\gamma_4 + (\gamma_0 - \gamma_3)\gamma_4 - 4i\lambda\tau\Big)\chi, \tag{2.4.17}$$

parameters $\theta,\ \tau$ being defined by the formulae

$$\begin{aligned}\tau^{2k} = {}& 32\lambda^2\tau^2(\bar{\chi}\chi) - 8i\lambda\tau[\bar{\chi}(\gamma_0 - \gamma_3)\chi]\\ &-32i\lambda^3\tau^3[\bar{\chi}(\gamma_0 + \gamma_3)\gamma_4\chi], \quad \theta = -\lambda^2\tau^2.\end{aligned} \tag{2.4.18}$$

Equation 8 from (2.3.5) by virtue of the change of variables

$$\varphi(\omega) = \omega^{-1/4}\phi(\omega),$$

where $\phi(\omega)$ is a new unknown four-component function, is reduced to the following system of ODEs:

$$2i\omega^{1/2}\gamma_2\dot{\phi} = \lambda\omega^{-1/4k}(\bar{\phi}\phi)^{1/2k}\phi.$$

Multiplying both parts of the above equality by $(i/2)\gamma_2\omega^{-1/2}$ we come to the equation

$$\dot{\phi} = (i\lambda/2)\omega^{-(1+2k)/4k}\gamma_2(\overline{\phi}\phi)^{1/2k}\phi, \tag{2.4.19}$$

the conjugate spinor satisfying the following equation:

$$\dot{\bar{\phi}} = -(i\lambda/2)\omega^{-(1+2k)/4k}\gamma_2(\bar{\phi}\phi)^{1/2k}\phi. \tag{2.4.20}$$

Multiplying (2.4.19) by $\bar{\phi}$ on the left, (2.4.20) by ϕ on the right and summing the equalities obtained we get

$$\dot{\bar{\phi}}\phi + \bar{\phi}\dot{\phi} = 0,$$

whence $\bar{\phi}\phi = C = \text{const}$. Consequently, equation (2.4.19) is equivalent to the linear ODE

$$\dot{\phi} = (i\lambda/2)\omega^{-(2k+1)/4k}C^{1/2k}\gamma_2\phi \tag{2.4.21}$$

which is supplemented by the additional constraint $\bar{\phi}\phi = C$.

Integration of (2.4.21) yields

$$\begin{aligned} k \neq 1/2, \quad \phi(\omega) &= \exp\{2i\lambda k(1-2k)^{-1}C^{1/2k}\gamma_2\omega^{(2k-1)/4k}\}\chi, \\ k = 1/2, \quad \phi(\omega) &= \exp\{(i\lambda/2)C\gamma_2 \ln\omega\}. \end{aligned}$$

Since $C = \bar{\phi}\phi = \bar{\chi}\chi$, the general solution of the initial equation 8 is given by the formulae

$$\begin{aligned} k \neq 1/2, \quad \varphi(\omega) &= \omega^{-1/4}\exp\{2i\lambda k(1-2k)^{-1}(\bar{\chi}\chi)^{1/2k} \\ &\quad \times\gamma_2\omega^{(2k-1)/4k}\}\chi, \\ k = 1/2, \quad \varphi(\omega) &= \omega^{-1/4}\exp\{(i\lambda/2)(\bar{\chi}\chi)\gamma_2\ln\omega\}\chi. \end{aligned} \tag{2.4.22}$$

To integrate the system of ODEs 19 from (2.3.5) we make the change of variables $\varphi(\omega) = \omega^{-1/4}\phi(\omega)$ transforming it to the form

$$2\omega^{1/2}\gamma_2\dot{\phi} + (1/2)(\gamma_0+\gamma_3)\phi = -i\lambda\omega^{-1/4k}(\bar{\phi}\phi)^{1/2k}\phi. \tag{2.4.23}$$

Solutions of the above system of ODEs satisfy the condition $\bar{\phi}\phi = C =$ const, whence it follows that equation (2.4.23) is linearized

$$2\omega^{1/2}\gamma_2\dot{\phi} + (1/2)(\gamma_0+\gamma_3)\phi = -i\lambda C^{1/2k}\omega^{-1/4k}\phi. \tag{2.4.24}$$

A general solution of (2.4.24) is looked for in the form

$$\begin{aligned} \phi(\omega) &= \{f_1(\omega) + \gamma_2 f_2(\omega) + (\gamma_0+\gamma_3)f_3(\omega) \\ &\quad +\gamma_2(\gamma_0+\gamma_3)f_4(\omega)\}\chi, \end{aligned} \tag{2.4.25}$$

where $f_i(\omega)$ are some real-valued scalar functions.

Substituting (2.4.25) into (2.4.24) we arrive at the following system of four linear ODEs:

$$\begin{aligned} &2\omega^{1/2}\dot{f}_1 = -i\lambda C^{1/2k}\omega^{-1/4k}f_2, \\ &2\omega^{1/2}\dot{f}_2 = i\lambda C^{1/2k}\omega^{-1/4k}f_1, \\ &2\omega^{1/2}\dot{f}_3 = (1/2)f_2 - i\lambda C^{1/2k}\omega^{-1/4k}f_4, \\ &2\omega^{1/2}\dot{f}_4 = (1/2)f_1 + i\lambda C^{1/2k}\omega^{-1/4k}f_3. \end{aligned}$$

Integration of the above system is carried out by standard methods. As a result, we have

$$f_1 = \cosh f(\omega), \quad f_2 = i\sinh f(\omega),$$

$$\begin{aligned} f_3 &= (i/4)\Big\{\cosh f(\omega) \int^{\omega} z^{-1/2} \sinh[2f(z)]dz \\ &\quad - \sinh f(\omega) \int^{\omega} z^{-1/2} \cosh[2f(z)]dz\Big\}, \qquad (2.4.26) \\ f_4 &= (1/4)\Big\{\cosh f(z) \int^{\omega} z^{-1/2} \cosh[2f(z)]dz \\ &\quad - \sinh f(\omega) \int^{\omega} z^{-1/2} \sinh[2f(z)]dz\Big\}, \end{aligned}$$

where

$$f(\omega) = \begin{cases} (\lambda C/2)\ln\omega, & k = 1/2, \\ 2\lambda k(1-2k)^{-1} C^{1/2k} \omega^{(2k-1)/4k}, & k \neq 1/2. \end{cases} \qquad (2.4.27)$$

From (2.4.26), (2.4.27) it follows that $\bar{\phi}\phi = \bar{\chi}\chi$, whence we conclude that $C = \bar{\chi}\chi$. Thus, the general solution of the system of ODEs 19 from (2.3.5) is given by the formula

$$\varphi(\omega) = \omega^{-1/4}\{f_1 + \gamma_2 f_2 + (\gamma_0+\gamma_3)f_3 + \gamma_2(\gamma_0+\gamma_3)f_4\}\chi,$$

functions $f_1(\omega), \ldots, f_2(\omega)$ being defined by (2.4.26), (2.4.27) with $C = \bar{\chi}\chi$.

In addition, we have succeeded in integrating the systems of ODEs numbered by 4, 24, 27 (with $\alpha = 0$). These systems can be written as follows:

$$(N/2)(\gamma_0+\gamma_3)\varphi + \Big(\omega(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3\Big)\dot{\varphi} = -i\lambda(\bar{\varphi}\varphi)^{1/2k}\varphi,$$

where cases $N = 1, 2, 3$ correspond to equations $4, 24, 27$ (with $\alpha = 0$) from (2.3.5).

Multiplying both parts of the above equality by the matrix $\omega(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3$ on the left yields

$$4\omega\dot{\varphi} = -\Big\{N(1+\gamma_0\gamma_3) + i\lambda(\bar{\varphi}\varphi)^{1/2k}\Big(\omega(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3\Big)\Big\}\varphi, \qquad (2.4.28)$$

the equation for the conjugate spinor taking the form

$$4\omega\dot{\bar{\varphi}} = -\bar{\varphi}\Big\{N(1-\gamma_0\gamma_3) - i\lambda(\bar{\varphi}\varphi)^{1/2k}\Big(\omega(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3\Big)\Big\}. \qquad (2.4.29)$$

Multiplying (2.4.28) by $\bar{\varphi}$, (2.4.29) by φ and summing we arrive at the relation

$$\dot{\bar{\varphi}}\varphi + \bar{\varphi}\dot{\varphi} = -2N\bar{\varphi}\varphi,$$

whence it follows that $\bar{\varphi}\varphi = C\omega^{-N/2}$, $C = \text{const}$.

Substitution of the result obtained into (2.4.28) gives rise to a linear equation for $\varphi(\omega)$

$$4\omega\dot{\varphi} = \{-N(1+\gamma_0\gamma_3) + i\tau\omega^{\theta}(\omega(\gamma_0+\gamma_3) + \gamma_0 - \gamma_3)\}\varphi,$$

where $\tau = -\lambda C^{1/2k}$, $\theta = -N/4k$.

Writing this equation component-wise (we assume that γ-matrices are of the form (1.1.8)) we get a system of four ODEs

$$\begin{aligned} &2\omega\dot{\varphi}^0 = i\tau\omega^{\theta+1}\varphi^2, \quad 2\omega\dot{\varphi}^1 = -N\varphi^1 + i\tau\omega^{\theta}\varphi^3,\\ &2\omega\dot{\varphi}^3 = i\tau\omega^{\theta+1}\varphi^2, \quad 2\omega\dot{\varphi}^2 = -N\varphi^2 + i\tau\omega^{\theta}\varphi^0, \end{aligned} \tag{2.4.30}$$

which is equivalent to the following second-order system of ODEs:

$$\begin{aligned} &\omega^2\ddot{\varphi}^0 + (1/2)(N-2\theta)\omega\dot{\varphi}^0 + (\tau^2/4)\omega^{2\theta+1}\varphi^0 = 0,\\ &\omega^2\ddot{\varphi}^3 + (1/2)(N-2\theta)\omega\dot{\varphi}^3 + (\tau^2/4)\omega^{2\theta+1}\varphi^3 = 0,\\ &\varphi^1 = -(2i/\tau)\omega^{-\theta}\dot{\varphi}^3, \quad \varphi^2 = -(2i/\tau)\omega^{-\theta}\dot{\varphi}^0. \end{aligned}$$

The first two equations of the above system are the Bessel-type equations [26, 197, 282]. Provided $\theta \neq -1/2$, their general solutions are given by the formulae

$$\begin{aligned} \varphi^0 &= \omega^{(2+2\theta-N)/4}\Big(\chi^0 J_\nu(z) + \chi^2 Y_\nu(z)\Big),\\ \varphi^3 &= \omega^{(2+2\theta-N)/4}\Big(\chi^1 Y_\nu(z) + \chi^3 J_\nu(z)\Big), \end{aligned} \tag{2.4.31}$$

where J_ν, Y_ν are the Bessel functions, $z = \tau(2\theta+1)^{-1}\omega^{(2\theta+1)/2}$, $\nu = (\theta + 1 - N/2)(1+2\theta)^{-1}$, $\chi^0, \dots, \chi^3$ are arbitrary complex constants. Consequently, the general solution of system of ODEs (2.4.30) is given by (2.4.31) and by the following formulae:

$$\begin{aligned} \varphi^2 &= \omega^{(2+2\theta-N)/4}\Big\{(i/2\tau)(N-2\theta-2)\omega^{-\theta-1}\\ &\quad\times\Big(\chi^0 J_\nu(z) + \chi^2 Y_\nu(z)\Big) - i\omega^{-1/2}\Big(\chi^0\dot{J}_\nu(z) + \chi^2\dot{Y}_\nu(z)\Big)\Big\},\\ \varphi^1 &= \omega^{(2+2\theta-N)/4}\Big\{(i/2\tau)(N-2\theta-2)\omega^{-\theta-1}\\ &\quad\times\Big(\chi^3 J_\nu(z) + \chi^1 Y_\nu(z)\Big) - i\omega^{-1/2}\Big(\chi^3\dot{J}_\nu(z) + \chi^1\dot{Y}_\nu(z)\Big)\Big\}. \end{aligned} \tag{2.4.32}$$

Formulae (2.4.31), (2.4.32) determine the general solution of nonlinear equations (2.4.28), (2.4.29) provided

$$\bar{\varphi}\varphi = \varphi^{0*}\varphi^2 + \varphi^0\varphi^{2*} + \varphi^{3*}\varphi^1 + \varphi^3\varphi^{1*} = C\omega^{-N/2}.$$

Substitution of expressions (2.4.31), (2.4.32) into this formula gives rise to the following equality:

$$2i\,(2\theta+1)(\tau\pi)^{-1}(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*}$$
$$-\chi^1\chi^{3*})\omega^{-N/2} = C\omega^{-N/2}$$

(we have used a well-known identity for the Bessel functions $J_\nu(z)\dot{Y}_\nu(z) - Y_\nu(z)\dot{J}_\nu(z) = 2(\pi z)^{-1}$ [282]).

Comparing the both parts of the above equality yields

$$C = 2i\,(2\theta+1)\,(\tau\pi)^{-1}(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*} - \chi^1\chi^{3*}),$$

whence

$$C = \{i\,(2k-N)\,(\pi k\lambda)^{-1}(\chi^{0*}\chi^2 - \chi^0\chi^{2*} + \chi^{3*}\chi^1 - \chi^3\chi^{1*})\}^{2k/(2k+1)}.$$

System (2.4.30) with $\theta = -1/2$ ($\Leftrightarrow\ k = -N/2$) is integrated in elementary functions. Omitting intermediate calculations we present the final result

1) $\tau^2 \neq N-1, \quad N = 2, 3$

$$\begin{aligned}
\varphi^0 &= \chi^0\omega^{\theta_+} + \chi^2\omega^{\theta_-},\\
\varphi^1 &= -(2i/\tau)\omega^{-1/2}(\theta_+\chi^3\omega^{\theta_+} + \theta_-\chi^1\omega^{\theta_-}),\\
\varphi^2 &= -(2i/\tau)\omega^{-1/2}(\theta_+\chi^0\omega^{\theta_+} + \theta_-\chi^2\omega^{\theta_-}),\\
\varphi^3 &= \chi^3\omega^{\theta_+} + \chi^1\omega^{\theta_-},
\end{aligned} \tag{2.4.33}$$

where $\theta_\pm = (1/4)\Big(1 - N \pm [(N-1)^2 - 4\tau^2]^{1/2}\Big)$, $\chi^0, \ldots, \chi^3$ are arbitrary constants; τ satisfies the equality

$$\begin{aligned}
&i(\chi^{0*}\chi^2 - \chi^0\chi^{2*} + \chi^{3*}\chi^1 - \chi^3\chi^{1*})\Big((N-1)^2 - 4\tau^2\Big)^{1/2}\\
&\quad = (-1)^{N+1}\tau^{N+1}\lambda^{-N};
\end{aligned}$$

2) $\tau \neq 0,\ N = 1$

$$\begin{aligned}
\varphi^0 &= \chi^0 \cos[(\tau/2)\ln\omega] + \chi^2 \sin[(\tau/2)\ln\omega],\\
\varphi^1 &= -i\omega^{-1/2}\Big(\chi^1 \cos[(\tau/2)\ln\omega] - \chi^3 \sin[(\tau/2)\ln\omega]\Big),\\
\varphi^2 &= -i\omega^{-1/2}\Big(\chi^2 \cos[(\tau/2)\ln\omega] - \chi^0 \sin[(\tau/2)\ln\omega]\Big),\\
\varphi^3 &= \chi^3 \cos[(\tau/2)\ln\omega] + \chi^1 \sin[(\tau/2)\ln\omega],
\end{aligned}\tag{2.4.34}$$

where $\chi^0, \ldots, \chi^3$ are arbitrary complex constants; τ is determined by the equality

$$\tau = -i\lambda(\chi^0\chi^{2*} - \chi^{0*}\chi^2 + \chi^3\chi^{1*} - \chi^1\chi^{3*});$$

3) $\tau = \varepsilon(N-1)/2, \quad \varepsilon = \pm 1$

$$\begin{aligned}
\varphi^0 &= \omega^{(1-N)/4}(\chi^0 + \chi^2 \ln\omega),\\
\varphi^1 &= (i/2\tau)(N-1)\omega^{-1/2}\varphi^3 + 4i\varepsilon(1-N)^{-1}\omega^{-(N+1)/4}\chi^1,\\
\varphi^2 &= (i/2\tau)(N-1)\omega^{-1/2}\varphi^0 + 4i\varepsilon(1-N)^{-1}\omega^{-(N+1)/4}\chi^2,\\
\varphi^3 &= \omega^{(1-N)/4}(\chi^3 + \chi^1 \ln\omega),
\end{aligned}\tag{2.4.35}$$

where $\chi^0, \ldots, \chi^3$ are complex constants satisfying the equality

$$2i(\chi^0\chi^{2*} - \chi^{0*} - \chi^{0*}\chi^2 + \chi^3\chi^{1*} - \chi^{3*}\chi^1) = (-1)^N\Big((N-1)/2\varepsilon\lambda\Big)^{N+1}.$$

Thus, the general solution of system (2.4.28) is given by formulae (2.4.31), (2.4.32) under $k \neq N/2$ and by formulae (2.4.33)–(2.4.35) under $k = N/2$.

Now we turn to Ansätze (2.3.16) which were obtained by reducing the nonlinear Dirac equation (2.2.1) by means of the one-parameter subgroups of the group $P(1,3)$ and then by means of symmetry groups of the reduced equations 5–7 from (2.3.3). As established in Section 2.3 Ansätze (2.3.16) reduce system of PDEs (2.4.1) to systems of ODEs (2.3.17) with $f_1 = \lambda(\bar\psi\psi)^{1/2k}$. Up to the sign at the nonlinear term $\lambda(\bar\varphi\varphi)^{1/2k}\varphi$, they coincide with systems of ODEs 1, 2, 8 from (2.3.5). Using this fact it is not difficult to construct their general solutions

$$\begin{aligned}
\varphi(\omega) &= \exp\{i\lambda\gamma_1(\bar\chi\chi)^{1/2k}\omega\}\chi,\\
\varphi(\omega) &= \omega^{-1/4}\begin{cases} \exp\Big\{2i\lambda k(1-2k)^{-1}(\bar\chi\chi)^{1/2k}\gamma_2 \\ \quad \times\, \omega^{(2k-1)/4k}\Big\}\chi, \quad k \neq 1/2,\\ \exp\{(i\lambda/2)(\bar\chi\chi)\gamma_2 \ln\omega\}\chi, \quad k = 1/2, \end{cases}
\end{aligned}$$

$$
\begin{aligned}
\varphi(\omega) &= \exp\{i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1\omega\}\chi,\\
\varphi(\omega) &= \omega^{-1/4}\exp\{-2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2\omega^{1/4}\}\chi,\\
\varphi(\omega) &= \exp\{i\lambda(\bar{\chi}\chi)^{1/2}\gamma_0\omega\}\chi,\\
\varphi(\omega) &= \exp\{-i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1\omega\}\chi,\\
\varphi(\omega) &= \omega^{-1/4}\exp\{2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2\omega^{1/4}\}\chi,\\
\varphi(\omega) &= \exp\{-i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1\omega\}\chi,\\
\varphi(\omega) &= \omega^{-1/4}\exp\{2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2\omega^{1/4}\}\chi.
\end{aligned}
\tag{2.4.36}
$$

Here χ is an arbitrary constant four-component column.

The fact that many of nonlinear systems (2.3.5) are integrable in quadratures is closely connected with their nontrivial symmetry. The last property, in its turn, is the consequence of the broad symmetry admitted by the initial PDE (Theorem 2.3.1). Therefore, the wider the symmetry group of the equation under study the more effective is the application of the group-theoretical methods for construction of its exact solutions.

1.2. *Exact solutions of equation (2.4.1).* Substitution of formulae (2.4.8), (2.4.9), (2.4.13), (2.4.15), (2.4.17), (2.4.22), (2.4.25)–(2.4.27), (2.4.31)–(2.4.36) into the corresponding $P(1,3)$-invariant Ansätze (2.2.8) and (2.3.16) yields the following classes of exact solutions of nonlinear spinor equation (2.4.1):

the case $k \in \mathbb{R}^1$

$$
\begin{aligned}
\psi_1(x) &= \exp\{-i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_0 x_0\}\chi,\\
\psi_2(x) &= \exp\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_3 x_3\}\chi,\\
\psi_3(x) &= \exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\\
&\quad\times\exp\{i\gamma_2[(\bar{\chi}\chi)^{1/2k}-(i/2)(\gamma_0+\gamma_3)]x_2\}\chi,\\
\psi_4(x) &= \exp\{-(1/2)(\gamma_0+\gamma_3)\gamma_1(x_0+x_3)\}\\
&\quad\times\exp\{(i\lambda/2)(\bar{\chi}\chi)^{1/2k}\gamma_1[2x_1+(x_0+x_3)^2]\}\chi,\\
\psi_5(x) &= \exp\{-(1/2)(\gamma_0+\gamma_3)\gamma_1(x_0+x_3)\}\exp\{(i\lambda/2)(1+\alpha^2)^{-1}\\
&\quad\times(\bar{\chi}\chi)^{1/2k}(\gamma_2-\alpha\gamma_1)[2(x_2-\alpha x_1)-\alpha(x_0+x_3)^2]\}\chi,\\
\psi_6(x) &= \exp\{(1/2)[x_1-\alpha\ln(x_0+x_3)](x_0+x_3)^{-1}(\gamma_0+\gamma_3)\gamma_1\}\\
&\quad\times\exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\exp\Big\{\Big(\gamma_2(\gamma_0+\gamma_3)+i\lambda(\bar{\chi}\chi)^{1/2k}\\
&\quad\times[\gamma_2-\beta(\gamma_0+\gamma_3)]\Big)[x_2-\beta\ln(x_0+x_3)]\Big\}\chi,\\
\psi_7(x) &= \exp\{(2\alpha)^{-1}(x_2+2\alpha\theta x_1)\gamma_0\gamma_3\}[(\theta\gamma_2-(2\alpha)^{-1}\gamma_1)\gamma_4-i\lambda\tau]\chi,\\
&\quad \alpha\in\mathbb{R}^1,\ \theta,\ \tau\ \text{ are determined by (2.4.11), (2.4.12);}
\end{aligned}
$$

$$
\begin{aligned}
\psi_8(x) &= \exp\{(2\alpha)^{-1}(2\alpha\theta x_3 - x_0)\gamma_1\gamma_2\}[(\theta\gamma_0 - (2\alpha)^{-1}\gamma_3)\gamma_4 - i\lambda\tau]\chi,\\
&\quad \alpha \in \mathbb{R}^1;\ \theta,\ \tau \text{ are determined by (2.4.14);}\\
\psi_9(x) &= \exp\{[(1/4)(x_3 - x_0) + \theta(x_0 + x_3)]\gamma_1\gamma_2\}\\
&\quad \times[4\theta(\gamma_0 + \gamma_3)\gamma_4 + (\gamma_0 - \gamma_3)\gamma_4 - 4i\lambda\tau]\chi,\\
&\quad \theta,\ \tau \text{ are determined by (2.4.18);}\\
\psi_{10}(x) &= \exp\{[-(1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2) + w_3\gamma_4](\gamma_0 + \gamma_3)\}\\
&\quad \times\exp\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1(x_1 + w_1)\}\chi;
\end{aligned}
$$

the case $k \in \mathbb{R}^1,\ k \neq 1/2$

$$
\begin{aligned}
\psi_{11}(x) &= \exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3)\}\varphi(x_0^2 - x_3^2),\\
&\quad \varphi(\omega) \text{ is determined by (2.4.31), (2.4.32) under } N = 1;\\
\psi_{12}(x) &= [(x_1 + w_1)^2 + (x_2 + w_2)^2]^{-1/4}\exp\{[-(1/2)(\dot{w}_1\gamma_1\\
&\quad +\dot{w}_2\gamma_2) + w_3\gamma_4](\gamma_0 + \gamma_3)\}\exp\{-(1/2)\gamma_1\gamma_2\\
&\quad \times\arctan[(x_1 + w_1)/(x_2 + w_2)]\}\exp\{2i\lambda k(1 - 2k)^{-1}\\
&\quad \times(\bar{\chi}\chi)^{1/2k}\gamma_2[(x_1 + w_1)^2 + (x_2 + w_2)^2]^{(2k-1)/4k}\}\chi,\\
\psi_{13}(x) &= (x_1^2 + x_2^2)^{-1/4}\exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3) - (1/2)\gamma_1\gamma_2\\
&\quad \times\arctan(x_1/x_2)\}\{f_1 + \gamma_2 f_2 + (\gamma_0 + \gamma_3)f_3 + \gamma_2(\gamma_0 + \gamma_3)f_4\}\chi,\\
&\quad f_i = f_i(x_1^2 + x_2^2) \text{ are determined by (2.4.26), (2.4.27)}\\
&\quad \text{under } k \neq 1/2;
\end{aligned}
$$

the case $k \in \mathbb{R}^1,\ k \neq 1$

$$
\begin{aligned}
\psi_{14}(x) &= \exp\{(1/2)x_1(x_0 + x_3)^{-1}(\gamma_0 + \gamma_3)\gamma_1\}\\
&\quad \times\exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3)\}\varphi(x_0^2 - x_1^2 - x_3^2),\\
&\quad \varphi(\omega) \text{ is determined by (2.4.31), (2.4.32) under } N = 2;
\end{aligned}
$$

the case $k \in \mathbb{R}^1,\ k \neq 3/2$

$$
\begin{aligned}
\psi_{15}(x) &= \exp\{(1/2)(x_0 + x_3)^{-1}(\gamma_0 + \gamma_3)(\gamma_1 x_1 + \gamma_2 x_2)\}\\
&\quad \times\exp\{(1/2)\gamma_0\gamma_3\ln(x_0 + x_3)\}\varphi(x \cdot x),\\
&\quad \varphi(\omega) \text{ is determined by (2.4.31), (2.4.32) under } N = 3;
\end{aligned}
$$

<u>the case $k = 1/2$</u>

$$
\begin{aligned}
\psi_{16}(x) &= \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi(x_0^2 - x_3^2),\\
&\quad \varphi(\omega) \text{ is determined by } (2.4.34);\\
\psi_{17}(x) &= (x_1^2 + x_2^2)^{-1/4} \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3) - (1/2)\gamma_1\gamma_2\\
&\quad \times \arctan(x_1/x_2)\}[f_1 + \gamma_2 f_2 + (\gamma_0 + \gamma_3)f_3 + \gamma_2(\gamma_0 + \gamma_3)f_4]\chi,\\
&\quad f_i = f_i(x_1^2 + x_2^2) \text{ are determined by } (2.4.26),\ (2.4.27)\\
&\quad \text{under } k = 1/2;\\
\psi_{18}(x) &= [(x_1 + w_1)^2 + (x_2 + w_2)^2]^{-1/4}\\
&\quad \times \exp\{[-(1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2) + w_3\gamma_4](\gamma_0 + \gamma_3)\}\\
&\quad \times \exp\{-(1/2)\gamma_1\gamma_2 \arctan[(x_1 + w_1)/(x_2 + w_2)]\}\\
&\quad \times \exp\{(i\lambda/2)(\bar{\chi}\chi)\gamma_2 \ln[(x_1 + w_1)^2 + (x_2 + w_2)^2]\}\chi;
\end{aligned}
$$

<u>the case $k = 1$</u>

$$
\begin{aligned}
\psi_{19}(x) &= w_0^{-1} \exp\Big\{\Big(-(1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2) + w_3\gamma_4\\
&\quad -(1/2)\dot{w}_0 w_0^{-1}[\gamma_1(x_1 + w_1) + \gamma_2(x_2 + w_2)]\Big)(\gamma_0 + \gamma_3)\Big\}\\
&\quad \times \exp\{i\lambda w_0^{-1}(\bar{\chi}\chi)^{1/2}\gamma_1(x_1 + w_1)\}\chi;\\
\psi_{20}(x) &= w_0^{-1/2}[(x_1 + w_1)^2 + (x_2 + w_2)^2]^{-1/4}\\
&\quad \times \exp\Big\{\Big(-(1/2)(\dot{w}_1\gamma_1 + \dot{w}_2\gamma_2) + w_3\gamma_4 - (1/2)\dot{w}_0 w_0^{-1}\\
&\quad \times[\gamma_1(x_1 + w_1) + \gamma_2(x_2 + w_2)](\gamma_0 + \gamma_3)\Big\}\\
&\quad \times \exp\{-(1/2)\gamma_1\gamma_2 \arctan[(x_1 + w_1)/(x_2 + w_2]\}\\
&\quad \times \exp\{-2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2[(x_1 + w_1)^2 + (x_2 + w_2)^2]^{1/4} w_0^{-1/2}\}\chi,\\
\psi_{21}(x) &= (\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2)(x_0^2 - x_1^2 - x_2^2)^{-3/2}\\
&\quad \times \exp\{i\lambda(\bar{\chi}\chi)^{1/2}\gamma_0 x_0(x_0^2 - x_1^2 - x_2^2)^{-1}\}\chi;\\
\psi_{22}(x) &= (\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2)(x_0^2 - x_1^2 - x_2^2)^{-3/2}\\
&\quad \times \exp\{-i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1 x_1(x_0^2 - x_1^2 - x_2^2)^{-1}\}\chi;\\
\psi_{23}(x) &= (\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2)(x_0^2 - x_1^2 - x_2^2)^{-1}(x_1^2 + x_2^2)^{-1/4}\\
&\quad \times \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\} \exp\{2i\lambda(\bar{\chi}\chi)^{1/2}\\
&\quad \times\gamma_2(x_1^2 + x_2^2)^{1/4}(x_0^2 - x_1^2 - x_2^2)^{-1/2}\}\chi;\\
\psi_{24}(x) &= \gamma_a x_a(x_b x_b)^{-3/2} \exp\{-i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1 x_1(x_a x_a)^{-1}\}\chi;
\end{aligned}
$$

$$\begin{aligned}
\psi_{25}(x) &= \gamma_a x_a (x_a x_a)^{-1}(x_1^2+x_2^2)^{-1/4}\exp\{-(1/2)\gamma_1\gamma_2 \\
&\quad \times \arctan(x_1/x_2)\}\exp\{2i\lambda(\bar\chi\chi)^{1/2}(x_1^2+x_2^2)^{1/4}(x_a x_a)^{-1/2}\}\chi; \\
\psi_{26}(x) &= \exp\{(1/2)x_1(x_0+x_3)^{-1}(\gamma_0+\gamma_3)\gamma_1\} \\
&\quad \times \exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\varphi(x_0^2-x_1^2-x_3^2), \\
&\quad \varphi(\omega) \text{ is determined by (2.4.33) or (2.4.35)} \\
&\quad \text{under } N=2;
\end{aligned}$$

the case $k = 3/2$

$$\begin{aligned}
\psi_{27}(x) &= \exp\{(1/2)(\gamma_0+\gamma_3)(\gamma_1 x_1+\gamma_2 x_2)(x_0+x_3)^{-1}\} \\
&\quad \times \exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\varphi(x\cdot x), \\
&\quad \varphi(\omega) \text{ is determined by (2.4.33) or (2.4.35)} \\
&\quad \text{under } N=3.
\end{aligned}$$

In the above formulae $w_0,\ w_1,\ w_2,\ w_3$ are arbitrary smooth functions of x_0+x_3, an overdot denotes differentiation with respect to x_0+x_3.

In addition, in [152, 155] we have constructed two other classes of exact solutions of system of PDEs (2.4.1)

the case $k = 1/2$

$$\begin{aligned}
\psi_{28}(x) &= \omega^{-1}\exp\{(1/2)\gamma_1(\gamma_0+\gamma_3)(x_0+x_3)\}\Big\{[\gamma_2+\beta(\gamma_0+\gamma_3)] \\
&\quad \times[x_2+\beta(x_0+x_3)]+(1/2)\gamma_1[2x_1+(x_0+x_3)^2]\Big\} \\
&\quad \times\exp\Big\{i\lambda(\bar\chi\chi)(\beta_1^2+\beta_2^2)^{-1}\omega^{-1}\Big(\beta_1[\gamma_2+\beta(\gamma_0+\gamma_3)]+\beta_2\gamma_1\Big) \\
&\quad \Big(\beta_1[x_2+\beta(x_0+x_3)]+(\beta_2/2)[2x_1+(x_0+x_3)^2]\Big)\Big\}\chi;
\end{aligned}$$

the case $k\in\mathbb{R}^1,\ k<0$

$$\begin{aligned}
\psi_{29}(x) &= \exp\{(1/2)\gamma_1(\gamma_0+\gamma_3)(x_0+x_3)\}\Big\{\Big([\gamma_2+\beta(\gamma_0+\gamma_3)][x_2+\beta \\
&\quad \times(x_0+x_3)]+(1/2)\gamma_1[2x_1+(x_0+x_3)^2]\Big)f(\omega)+ig(\omega)\Big\}\chi.
\end{aligned}$$

Here $\alpha,\ \beta,\ \beta_1,\ \beta_2$ are arbitrary constants,

$$\omega = [x_2+\beta(x_0+x_3)]^2+(1/4)[2x_1+(x_0+x_3)^2]^2,$$
$$f(\omega) = |k|^{1/2}\Big(\varepsilon(1-k)^{1/2}\lambda^{-1}(\bar\chi\chi)^{-1/2k}\Big)^k\omega^{-(k+1)/2},$$
$$g(\omega) = -\varepsilon(1-k)^{1/2}\Big(\varepsilon(1-k)^{1/2}\lambda^{-1}(\bar\chi\chi)^{-1/2k}\Big)^k\omega^{-k/2},\quad \varepsilon=\pm1.$$

Thus, we have constructed wide classes of exact solutions of the nonlinear Dirac equation (2.41), some of them containing arbitrary functions. By a special choice of these functions we can select subclasses of exact solutions possessing important additional properties.

For example, if we put

$$w_0 = \exp\{\theta^2 (x_0 + x_3)^2\}, \quad \theta \in \mathbb{R}^1, \quad w_1 = w_2 = w_3 = 0$$

in the solution $\psi_{19}(x)$, then it takes the form

$$\begin{aligned} \psi(x) \quad = \quad & \exp\{-\theta^2 (x_0 + x_3)^2\}\Big(1 + \theta^2 (x_0 + x_3)(\gamma_1 x_1 + \gamma_2 x_2) \\ & \times(\gamma_0 + \gamma_3)\Big) \exp\Big\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1 x_1 \exp\{-\theta^2 (x_0 + x_3)^2\}\Big\}\chi. \end{aligned} \tag{2.4.37}$$

This solution is localized inside the infinite cylinder having the generatix parallel to the coordinate axis Ox_3. In addition, it decreases exponentially as $x_0 \to +\infty$.

It is worth noting that (2.4.37) under $\theta = 0$ becomes the plane-wave solution

$$\psi(x) = \exp\{i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1 x_1\}\chi. \tag{2.4.38}$$

Consequently, (2.4.37) can be considered as a perturbation of the stationary state (2.4.38).

1.3. *Generation of solutions.* Solutions $\psi_1(x)-\psi_{29}(x)$ depend on the variables x_μ in asymmetrical way, while in equation (2.4.1) all independent variables are enjoying equal rights. Using the language of physics we can say that system of PDEs (2.4.1) is solved in some fixed reference frame. To obtain solutions (more precisely, families of solutions) not depending on the choice of a reference frame it is necessary to apply the procedure of generating solutions by transformations from the Poincaré group [137, 139, 140, 155].

Let the equation under study be invariant under the Lie group of transformations of the form

$$x'_\mu = f_\mu(x, \theta), \quad \psi'(x') = A(x, \theta)\psi(x), \tag{2.4.39}$$

where $A(x, \theta)$ is an invertible $(m \times m)$-matrix, $\theta = (\theta_1,\ \theta_2, \ldots, \theta_r)$ are group parameters. In addition, there is some particular solution $\psi_I(x)$ of the equation considered.

Theorem 2.4.1. *The m-component function $\psi_{II}(x)$ determined by the equality*

$$\psi_{II} = A^{-1}(x, \theta)\psi_I\Big(f(x, \theta)\Big) \tag{2.4.40}$$

is a solution of PDE admitting the Lie group (2.4.39).

Proof. According to the definition of the invariance group, the Lie group (2.4.39) transforms the set of solutions of the equation under study into itself. In other words, provided $\psi = \psi(x)$ is a solution of the equation written in coordinates x, $\psi(x)$ the function constructed by means of formulae (2.4.39) is a solution of the same equation written in coordinates x', $\psi'(x')$. Resolving (2.4.39) with respect to $\psi(x)$ we have

$$\psi(x) = A^{-1}(x,\theta)\psi'(x'),$$

whence due to (2.4.39) we get

$$\psi(x) = A^{-1}(x,\theta)\psi'\Big(f(x,\theta)\Big).$$

Denoting $\psi = \psi_{II}$, $\psi' = \psi_I$ yields (2.4.40). $\triangleright$

Using Theorem 2.4.1 it is possible to obtain a r-parameter family of exact solutions starting from a single solution.

Definition 2.4.1. Formula (2.4.40) is called the formula for generating solutions by transformations from the group (2.4.39).

Definition 2.4.2. A family of solutions of the form

$$\begin{aligned}&\psi(x) = \psi_0(x,\tau), \quad \tau = (\tau_1, \tau_2, \ldots, \tau_s) \in \mathbb{R}^s,\\ &R_i(\tau) = 0, \quad i = 1, \ldots, s-n+1, \quad 1 \le n \le s\end{aligned}$$

is called G-ungenerable (or ungenerable) provided the equality

$$A^{-1}(x,\theta)\psi_0\Big(f(x,\theta),\tau\Big) = \psi_0\Big(x, \tau'(\tau,\theta)\Big)$$

holds and what is more $R_i(\tau'(\tau,\theta)) = 0$, $i = 1, \ldots, s-n+1$.

Using the final transformations from the group $C(1,3)$ (1.1.24)–(1.1.28) and Theorem 2.4.1 we obtain formulae of generating solutions by transformations from the conformal group $C(1,3)$.

1) the group of translations

$$\psi_{II}(x) = \psi_I(x'), \quad x'_\mu = x_\mu + \theta_\mu; \tag{2.4.41}$$

2) the Lorentz group $O(1,3)$

a) the group of rotations $O(3)$

$$\begin{aligned} &\psi_{II}(x) = \exp\{(1/2)\varepsilon_{abc}\theta_a S_{bc}\}\psi_I(x'),\\ &x'_0 = x_0, \quad x'_a = x_a\cos\theta - \theta^{-1}\varepsilon_{abc}\theta_b x_c \sin\theta\\ &\quad +\theta^{-2}\theta_a(\theta_b x_b)(1-\cos\theta); \end{aligned} \tag{2.4.42}$$

b) the Lorentz transformations

$$\begin{aligned} &\psi_{II}(x) = \exp\{-(\theta_0/2)\gamma_0\gamma_a\}\psi_I(x'),\\ &x'_0 = x_0\cosh\theta_0 + x_a\sinh\theta_0,\\ &x'_a = x_a\cosh\theta_0 + x_0\sinh\theta_0, \quad x'_b = x_b,\ b\neq a; \end{aligned} \tag{2.4.43}$$

3) the group of scale transformations

$$\psi_{II}(x) = e^{k\theta_0}\psi_I(x'), \quad x'_\mu = x_\mu e^{\theta_0}; \tag{2.4.44}$$

4) the group of special conformal transformations

$$\begin{aligned} &\psi_{II}(x) = \sigma^{-2}(x)(1-\gamma\cdot x\gamma\cdot\theta)\psi_I(x'),\\ &x'_\mu = (x_\mu - \theta_\mu x\cdot x)\sigma^{-1}(x). \end{aligned} \tag{2.4.45}$$

Here $\theta_0,\dots,\theta_3$ are real constants, $\theta = (\theta_a\theta_a)^{1/2}$, $\sigma(x) = 1-2\theta\cdot x+\theta\cdot\theta x\cdot x$.

As an example, we will consider the procedure of generating the solution $\psi_1(x)$. Let us apply formula (2.4.43) with $a=3$ to $\psi_1(x)$

$$\psi_{II}(x) = \exp\{-(\theta_0/2)\gamma_0\gamma_3\}\exp\{-i\lambda(\bar\chi\chi)^{1/2k}(x_0\cosh\theta_0 + x_3\sinh\theta_0)\gamma_0\}\chi.$$

We rewrite this expression as follows

$$\begin{aligned} \psi_{II}(x) = {}&\exp\{-(\theta_0/2)\gamma_0\gamma_3\}\Big\{\cos\Big(\lambda(\bar\chi\chi)^{1/2k}(x_0\cosh\theta_0\\ &+x_3\sinh\theta_0)\Big) - i\gamma_0\sin\Big(\lambda(\bar\chi\chi)^{1/2k}(x_0\cosh\theta_0 + x_3\sinh\theta_0)\Big)\Big\}\\ &\times\exp\{-(\theta_0/2)\gamma_0\gamma_3\}\chi. \end{aligned}$$

Taking into account the identities

$$V\gamma_\mu V^{-1} = \begin{cases} \gamma_0\cosh\theta_0 + \gamma_3\sinh\theta_0, & \mu = 0,\\ \gamma_3\cosh\theta_0 + \gamma_0\sinh\theta_0, & \mu = 3,\\ \gamma_\mu, & \mu = 1,2, \end{cases}$$

where $V = \exp\{-(\theta_0/2)\gamma_0\gamma_3\}$, which are proved with the help of the Campbell-Hausdorff formula [41, 179] we have

$$\psi_{II}(x) = \Big\{\cos\Big(\lambda(\bar\chi'\chi')^{1/2k}(x_0\cosh\theta_0 + x_3\sinh\theta_0)\Big) - i(\gamma_0\cosh\theta_0 + \gamma_3\sinh\theta_0)\sin\Big(\lambda(\bar\chi'\chi')^{1/2k}(x_0\cosh\theta_0 + x_3\sinh\theta_0)\Big)\Big\}\chi',$$

where $\chi' = \exp\{-(\theta_0/2)\gamma_0\gamma_3\}$.

Using formula (2.4.42) yields the following family of exact solutions:

$$\begin{aligned}\psi_{II}(x) &= \Big\{\cos\Big(\lambda(\bar\chi\chi)^{1/2k}a\cdot x\Big) - i\gamma\cdot a\sin\Big(\lambda(\bar\chi\chi)^{1/2k}a\cdot x\Big)\Big\}\chi \\ &= \exp\{-i\lambda(\bar\chi\chi)^{1/2k}(\gamma\cdot a)(a\cdot x)\}\chi,\end{aligned} \tag{2.4.46}$$

where a_μ are arbitrary real parameters satisfying the condition $a\cdot a = 1$.

It is not difficult to verify that family (2.4.46) is invariant with respect to transformations (2.4.41), (2.4.44).

The family of solutions (2.4.46) depends on the variables x_μ in symmetrical way. Let us show that it is invariant under the Lorentz group $O(1,3)$. Applying, for example, formula (2.4.42) to (2.4.46) and grouping terms in a proper way we arrive at the following family of solutions of PDE (2.4.1):

$$\psi_{II}(x) = \exp\{-i\lambda(\bar\chi'\chi')^{1/2k}(\gamma\cdot a')(a'\cdot x)\}\chi',$$

where

$$\begin{aligned}&a'_0 = a_0, \quad a'_b = a_b\cos\theta - \theta^{-1}\varepsilon_{bcd}a_c\theta_d + \theta^{-2}\theta_b(\theta_c a_c)(1-\cos\theta), \\ &\chi' = \exp\{(1/2)\varepsilon_{abc}\theta_a S_{bc}\}\chi.\end{aligned}$$

Since $a'\cdot a' = 1$, the obtained family coincides with (2.4.46). Thus, we have constructed the $\widetilde{P}(1,3)$-ungenerable family of exact solutions of the nonlinear Dirac equation. The transition from $\psi_1(x)$ to (2.4.46) seems to be of principal importance because we obtain the class of exact solutions having the same invariance group as the initial equation (2.4.1). In other words, the family of solutions (2.4.46) contains complete information about the Lie symmetry of the nonlinear Dirac equation (2.4.1).

Generating in the same way solutions $\psi_2(x)-\psi_6(x)$ we obtain the following $\widetilde{P}(1,3)$-ungenerable families of exact solutions of system of nonlinear PDEs (2.4.1):

$$\psi_2(x) \quad = \quad \exp\{i\lambda(\bar\chi\chi)^{1/2k}(\gamma\cdot b)(b\cdot x)\}\chi,$$

$$\begin{aligned}
\psi_3(x) &= \exp\{(1/2)(\gamma\cdot a)(\gamma\cdot d)\ln[\theta(a\cdot z+d\cdot z)]\}\\
&\quad\times\exp\{i\gamma\cdot c[(\bar\chi\chi)^{1/2k}-(i/2)(\gamma\cdot a+\gamma\cdot d)]c\cdot z\}\chi,\\
\psi_4(x) &= \exp\{-(\theta/2)(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\}\\
&\quad\times\exp\{(i\lambda/2)(\bar\chi\chi)^{1/2k}(\gamma\cdot b)[2b\cdot z+\theta(a\cdot z+d\cdot z)^2]\}\chi,\\
\psi_5(x) &= \exp\{-(\theta/2)(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\}\\
&\quad\times\exp\{(i\lambda/2)(1+\alpha^2)^{-1}(\bar\chi\chi)^{1/2k}(\gamma\cdot c-\alpha\gamma\cdot b)\\
&\quad\times[2(c\cdot z-\alpha b\cdot z)-\alpha\theta(a\cdot z+d\cdot z)^2]\}\chi,\\
\psi_6(x) &= \exp\Big\{(2\theta)^{-1}\Big(\theta b\cdot z-\alpha\ln[\theta(a\cdot z+d\cdot z)]\Big)(a\cdot z+d\cdot z)^{-1}\\
&\quad\times(\gamma\cdot a+\gamma\cdot d)\gamma\cdot b\Big\}\exp\{(1/2)(\gamma\cdot a)(\gamma\cdot d)\ln[\theta(a\cdot z+d\cdot z)]\Big\}\\
&\quad\times\exp\Big\{\Big(\gamma\cdot c(\gamma\cdot a+\gamma\cdot d)+i\lambda(\bar\chi\chi)^{1/2k}[\gamma\cdot c-\beta(\gamma\cdot a+\gamma\cdot d)]\Big)\\
&\quad\times\Big(c\cdot z-\beta\theta^{-1}\ln[\theta(a\cdot z+d\cdot z)]\Big)\Big\}\chi,
\end{aligned}$$

where $z_\mu=x_\mu+\theta_\mu$; $\alpha,\ \beta,\ \theta,\ \theta_\mu$ are arbitrary constants.

Hereafter we denote by $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu,\ \mu=0,\dots,3$ arbitrary real constants satisfying the following conditions:

$$\begin{aligned}
&a\cdot a=-b\cdot b=-c\cdot c=-d\cdot d=1,\\
&a\cdot b=a\cdot c=a\cdot d=b\cdot c=b\cdot d=c\cdot d=0.
\end{aligned}\tag{2.4.47}$$

Evidently, the four-vectors with components $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ form a basis in the Minkowski space $R(1,3)$ with the scalar product $x\cdot y=x_\mu y^\mu$.

Provided the parameter k in (2.4.1) is equal to 3/2, this equation admits the conformal group $C(1,3)$. Consequently, we can generate solutions by transformations (2.4.45). Let us give an example of the $C(1,3)$-ungenerable family of exact solutions of the conformally-invariant Dirac-Gürsey equation

$$\begin{aligned}
\psi(x) &= \sigma^{-2}(x)(1-\gamma\cdot x\gamma\cdot\theta)\exp\{-i\lambda(\bar\chi\chi)^{1/2k}(\gamma\cdot a)\\
&\quad\times(a\cdot x-a\cdot\theta x\cdot x)\sigma^{-1}(x)\}\chi.
\end{aligned}$$

2. $\widetilde{P}(1,3)$-invariant solutions of the nonlinear Dirac equation (2.4.1). Now we turn to reduced equations (2.3.22) putting $R=\lambda(\bar\varphi\varphi)^{1/2k}\varphi$. To integrate these we need some well-known facts from the general theory of systems of linear ODEs.

Definition 2.4.3. By a normalized solution of the system of linear ODEs

$$\dot\varphi(\omega)=B(\omega)\varphi(\omega)\tag{2.4.48}$$

we mean the (4×4)-matrix $\Omega^{\omega}_{\omega_0}(B)$ satisfying the following conditions:

$$\frac{d\Omega^{\omega}_{\omega_0}}{d\omega} = B(\omega)\Omega^{\omega}_{\omega_0}, \quad \Omega^{\omega_0}_{\omega_0} = I,$$

where $\omega_0 = \mathrm{const}$, I is the unit (4×4)-matrix.

The normalized solution of system (2.4.48) is given by the following infinite series [197]:

$$\Omega^{\omega}_{\omega_0} = I + \int\limits_{\omega_0}^{\omega} B(\tau)d\tau + \int\limits_{\omega_0}^{\omega} B(\tau) \int\limits_{\omega_0}^{\tau} B(\tau_1)d\tau_1 d\tau + \ldots \tag{2.4.49}$$

If we succeed in constructing the normalized solution of system of ODEs (2.4.48) in explicit form, then its general solution is given by the formulae

$$\varphi(\omega) = \Omega^{\omega}_{\omega_0}(B)\chi, \quad \varphi(\omega_0) = \chi,$$

where χ is an arbitrary constant four-component column.

We will consider in detail a procedure of integration of system of ODEs 1 from (2.3.22). On multiplying it by the matrix $(i/2)\gamma_3$ on the left we get

$$\dot{\varphi} = (1/8)(2k-1)(1+\gamma_0\gamma_3)\varphi + (i\lambda/2)\gamma_3(\bar{\varphi}\varphi)^{1/2k}\varphi, \tag{2.4.50}$$

while the conjugate spinor $\bar{\varphi}(\omega)$ satisfies system of ODEs of the form

$$\dot{\bar{\varphi}} = (1/8)(2k-1)\bar{\varphi}(1-\gamma_0\gamma_3) - (i\lambda/2)\bar{\varphi}\gamma_3(\bar{\varphi}\varphi)^{1/2k}. \tag{2.4.51}$$

Multiplying (2.4.50) by $\bar{\varphi}$ on the left, (2.4.51) by φ on the right and summing we come to the linear ODE for $\bar{\varphi}\varphi$

$$\bar{\varphi}\dot{\varphi} + \dot{\bar{\varphi}}\varphi = (\bar{\varphi}\varphi)^{\cdot} = (1/4)(2k-1)\bar{\varphi}\varphi,$$

which general solution reads

$$\bar{\varphi}\varphi = C\exp\{(1/4)(2k-1)\omega\}, \quad C \in \mathbb{R}^1. \tag{2.4.52}$$

Substitution of (2.4.52) into (2.4.50) gives rise to the system of linear ODEs

$$\begin{aligned} \dot{\varphi} \;\; = \;\; & \Big\{(1/8)\,(2k-1)\,(1+\gamma_0\gamma_3) + (i\lambda/2)C^{1/2k} \\ & \times \exp\{(2k-1)(8k)^{-1}\omega\}\gamma_3\Big\}\varphi. \end{aligned} \tag{2.4.53}$$

Let Ω_0^ω be a normalized solution of system (2.4.52). Then the general solution of (2.4.52) is given by the formula

$$\varphi(\omega) = \Omega_0^\omega \chi. \tag{2.4.54}$$

Substituting (2.4.54) into (2.4.52) we have

$$\bar{\chi}\bar{\Omega}_0^\omega \Omega_0^\omega \chi = C \exp\{(1/4)(2k-1)\omega\},$$

where $\overline{\Omega}_0^\omega = \gamma_0 (\Omega_0^\omega)^\dagger \gamma_0$. As $\Omega_0^\omega|_{\omega=0} = I$ and $\bar{\Omega}_0^\omega|_{\omega=0} = I$, from the above relation it follows that $\bar{\chi}\chi = C$. Inserting $C = \bar{\chi}\chi$ into (2.4.54) we obtain the general solution of the initial system of nonlinear ODEs (2.4.50).

Substitution of the result obtained into the corresponding $\widetilde{P}(1,3)$-invariant Ansatz gives rise to the exact solution of the nonlinear spinor equation (2.4.1)

$$\psi(x) = \exp\{(1/4)(\gamma_0\gamma_3 - 2k)\ln(x_0 + x_3)\}\Omega_0^\omega \chi, \tag{2.4.55}$$

where $\omega = \ln(x_0 + x_3) - x_0 + x_3$.

In a similar way we can integrate systems of ODEs 2, 4, 7, 9, 11, 13, 16, 17, 20, 23–25 from (2.3.22) (a detailed analysis of these equations has been performed in [8]). Here only the cases, when infinite series (2.4.49) can be summed up, are considered.

If we put in (2.4.53) $k = 1/2$, then a system of linear ODEs with constant coefficients is obtained. Its general solution has the form (2.4.54), where

$$\begin{aligned} \Omega_0^\omega &= I + \int_0^\omega B d\tau + \int_0^\omega B \int_0^\tau B d\tau_1 d\tau + \ldots \\ &= I + \omega B + (2!)^{-1}\omega^2 B^2 + (3!)^{-1}\omega^3 B^3 + \ldots \\ &= \exp\{B\omega\}. \end{aligned} \tag{2.4.56}$$

In (2.4.56) $B = (i\lambda/2)(\bar{\chi}\chi)\gamma_3$.

Substitution of (2.4.56) into (2.4.55) gives rise to the exact solution of system of nonlinear PDEs (2.4.1) with $k = 1/2$

$$\begin{aligned} \psi(x) &= \exp\{(1/4)(\gamma_0\gamma_3 - 1)\ln(x_0 + x_3)\} \\ &\quad \times \exp\{(i\lambda/2)(\bar{\chi}\chi)\gamma_3[\ln(x_0 + x_3) - x_0 + x_3]\}\chi. \end{aligned} \tag{2.4.57}$$

Similarly, computing the normalized solutions of systems of ODEs 4 (under $\alpha = 0, k = 1/2$), 9 (under $k = 3/2$), 11 (under $k = 5/2$), 13 (under $k = 1/2$),

20, 22 from (2.3.22) we get their general solutions in the form (2.4.54)

$$
\begin{aligned}
\varphi(\omega) &= \exp\{(i\lambda/2)(\bar{\chi}\chi)(\gamma_3 - \gamma_0 + 2\gamma_1)\omega\}\chi,\\
\varphi(\omega) &= \exp\{-(i\lambda/2)(\bar{\chi}\chi)^{1/3}\gamma_0\omega\}\chi,\\
\varphi(\omega) &= \exp\{-(i\lambda/2)(\bar{\chi}\chi)^{1/5}\gamma_0\omega\}\chi,\\
\varphi(\omega) &= \exp\{i\lambda(1+\alpha^2)^{-1}(\alpha\gamma_2 - \gamma_1)(\bar{\chi}\chi)\omega\}\chi,\\
\varphi(\omega) &= \exp\{[(1/2\alpha)(2k-1)(\gamma_0-\gamma_3)\gamma_1\\
&\quad -i\lambda\alpha^{-2}(\gamma_0-\gamma_3+\alpha\gamma_1)(\bar{\chi}\chi)^{1/2k}]\omega\}\chi,\\
\varphi(\omega) &= \exp\Big\{\Big((1/2\beta)(1+\beta^2)^{-1}[(2k\beta^2-1)\gamma_1\\
&\quad -\beta(2k+1)\gamma_2](\gamma_0-\gamma_3) - i\lambda(1+\beta^2)^{-1}\\
&\quad \times[\gamma_2 - \beta\gamma_1 - (\beta/\alpha)(\gamma_0-\gamma_3)](\bar{\chi}\chi)^{1/2k}]\omega\Big\}\chi.
\end{aligned}
$$

We have also succeeded in integrating systems of ODEs 30, 37, 41. They can be represented in the following unified form:

$$
i\Big(\gamma_2 - (\gamma_0-\gamma_3)z\Big)\frac{d\varphi}{dz} = \Big(i\theta(\gamma_0-\gamma_3) + \lambda(\bar{\varphi}\varphi)^{1/2k}\Big)\varphi, \tag{2.4.58}
$$

where the case $\theta = k,\ z = \omega$ corresponds to the system 30, the case $\theta = (1/2)(1-2k),\ z = \omega$ to the system 37 and the case $\theta = k,\ z = \omega - 1$ to the system 41.

Rewrite equation (2.4.58) in the equivalent form

$$
\dot{\varphi} = \Big\{\theta\gamma_2(\gamma_0-\gamma_3) - i\lambda(\bar{\varphi}\varphi)^{1/2k}\Big(\gamma_2 - (\gamma_0-\gamma_3)z\Big)\Big\}\varphi.
$$

Since $\bar{\varphi}\varphi \equiv \tau = \text{const}$, the above equation is linearized

$$
\dot{\varphi} = \Big\{\theta\gamma_2(\gamma_0-\gamma_3) - i\lambda\tau^{1/2k}\Big(\gamma_2 - (\gamma_0-\gamma_3)z\Big)\Big\}\varphi. \tag{2.4.59}
$$

The general solution of the system of ODEs (2.4.59) can be represented in the form

$$
\varphi = \Big(f_1(z) + f_2(z)\gamma_2 + f_3(z)(\gamma_0-\gamma_3) + f_4(z)\gamma_2(\gamma_0-\gamma_3)\Big)\chi, \tag{2.4.60}
$$

where χ is an arbitrary four-component constant column and functions $f_1, \ldots,$ f_4 satisfy the following system of ODEs:

$$
\begin{aligned}
&\dot{f}_1 = i\lambda\tau^{1/2k}f_2, \quad \dot{f}_2 = -i\lambda\tau^{1/2k}f_1,\\
&\dot{f}_3 = i\lambda\tau^{1/2k}f_4 + \theta f_2 + i\lambda\tau^{1/2k}zf_1,\\
&\dot{f}_4 = -i\lambda\tau^{1/2k}f_3 + \theta f_1 - i\lambda\tau^{1/2k}zf_2.
\end{aligned}
$$

The above system is integrated by the standard methods, its particular solution reads

$$
\begin{aligned}
&f_1 = \cosh(\lambda\tau^{1/2k}z), \quad f_2 = -i\sinh(\lambda\tau^{1/2k}z), \\
&f_3 = -\Big((2\theta+1)i/4\lambda\Big)\tau^{-1/2k}\cosh(\lambda\tau^{1/2k}z) + (i/2)(1+z)\sinh(\lambda\tau^{1/2k}z), \\
&f_4 = \Big((2\theta+1)/4\lambda\Big)\tau^{-1/2k}\sinh(\lambda\tau^{1/2k}z) + (1/2)(1-z)\cosh(\lambda\tau^{1/2k}z).
\end{aligned}
\tag{2.4.61}
$$

As a direct computation shows, the function (2.4.60) satisfies an identity

$$
\begin{aligned}
\bar{\varphi}\varphi \;=\; & \bar{\chi}\Big(|f_1|^2 - |f_2|^2 + (f_1 f_2^* + f_1^* f_2)\gamma_2 + (f_1^* f_3 + f_1 f_3^* - f_2^* f_4 \\
& - f_2 f_4^*)(\gamma_0 - \gamma_3) + (f_1^* f_4 - f_1 f_4^* + f_2^* f_3 - f_2 f_3^*)\gamma_2(\gamma_0 - \gamma_3)\Big)\chi.
\end{aligned}
$$

Substituting into its right-hand side formulae (2.4.61) we get

$$
\tau = \bar{\varphi}\varphi = \bar{\chi}\chi. \tag{2.4.62}
$$

Consequently, we have established that the general solution of the system of nonlinear ODEs (2.4.58) is given by the formulae (2.4.60)–(2.4.62).

Substitution of the expressions obtained above into the corresponding $\widetilde{P}(1,3)$-invariant Ansätze (2.2.8) yields the following exact solutions of the nonlinear Dirac equation (2.4.1):

<u>the case $k = 1/2$</u>

$$
\begin{aligned}
\psi(x) \;=\; & \exp\{(1/2)\gamma_1\gamma_2 \arctan(x_2/x_1) + (1/4)(\gamma_0\gamma_3 - 1)\ln(x_1^2 + x_2^2)\} \\
& \times \exp\{(i\lambda/4)(\gamma_3 - \gamma_0 + 2\gamma_1)(\bar{\chi}\chi)[x_0 - x_3 - \ln(x_1^2 + x_2^2)]\}\chi; \\
\psi(x) \;=\; & \exp\{(1/2)\gamma_1\gamma_2 \arctan(x_2/x_1) - (1/4)\ln(x_1^2 + x_2^2)\} \\
& \times \exp\{i\lambda(1+\alpha^2)^{-1}(\bar{\chi}\chi)(\alpha\gamma_2 - \gamma_1)[\alpha \arctan(x_2/x_1) \\
& - (1/2)\ln(x_1^2 + x_2^2)]\}\chi;
\end{aligned}
$$

<u>the case $k = 3/2$</u>

$$
\begin{aligned}
\psi(x) \;=\; & \exp\{-(1/2)\gamma_1(\gamma_0 - \gamma_3)x_1(x_0 - x_3)^{-1}\} \\
& \times \exp\{-(1/4)(\gamma_0\gamma_3 + 3)\ln(x_0 - x_3)\}\exp\{-(i\lambda/2)(\bar{\chi}\chi)^{1/3} \\
& \times \gamma_0[(x_0^2 - x_1^2 - x_3^2)(x_0 - x_3)^{-1} + \ln(x_0 - x_3)]\}\chi,
\end{aligned}
$$

<u>the case $k = 5/2$</u>

$$
\begin{aligned}
\psi(x) \;=\; & \exp\{-(1/2)\gamma_1(\gamma_0 - \gamma_3)x_1(x_0 - x_3)^{-1} - (1/2)\gamma_2(\gamma_0 - \gamma_3) \\
& \times x_2(x_0 - x_3)^{-1}\}\exp\{-(1/4)(\gamma_0\gamma_3 + 5)\ln(x_0 - x_3)\} \\
& \times \exp\{-(i\lambda/2)(\bar{\chi}\chi)^{1/5}\gamma_0[x \cdot x(x_0 - x_3)^{-1} + \ln(x_0 - x_3)]\}\chi;
\end{aligned}
$$

the case of arbitrary k

$$
\begin{aligned}
\psi(x) &= \exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1} - k\ln(x_0-x_3)\}\\
&\quad \times \exp\{[(1/2\alpha)\gamma_1(\gamma_0-\gamma_3)(1-2k) - i\lambda\alpha^{-2}(\gamma_0-\gamma_3\\
&\quad +\alpha\gamma_1)(\bar\chi\chi)^{1/2k}][\ln(x_0-x_3)+\alpha x_1(x_0-x_3)^{-1}])\}\chi;\\
\psi(x) &= \exp\{(1/2\beta)\gamma_1(\gamma_0-\gamma_3)(x_2-\beta x_1)(x_0-x_3)^{-1}\\
&\quad -k\ln(x_0-x_3)\}\exp\Big\{\Big((1/2\beta)(1+\beta^2)^{-1}[(2\beta^2k+1)\gamma_1\\
&\quad -\beta(2k+1)\gamma_2](\gamma_0-\gamma_3) - i\lambda(1+\beta^2)^{-1}[\gamma_2-\beta\gamma_1\\
&\quad -(\beta/\alpha)(\gamma_0-\gamma_3)](\bar\chi\chi)^{1/2k}\Big)[(x_2-\beta x_1)(x_0-x_3)^{-1}\\
&\quad -(\beta/\alpha)\ln(x_0-x_3)]\}\chi;\\
\psi(x) &= (x_0-x_3)^{-k}\Big(f_1+f_2\gamma_2+f_3(\gamma_0-\gamma_3)+f_4\gamma_2(\gamma_0-\gamma_3)\Big)\chi,\\
&\quad \text{where } f_i=f_i[x_2(x_0-x_3)^{-1}] \text{ are determined by (2.4.61),}\\
&\quad \text{(2.4.62) with } \theta=k;\\
\psi(x) &= (x_0-x_3)^{-k}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)x_1(x_0-x_3)^{-1}\}\\
&\quad \times\Big(f_1+f_2\gamma_2+f_3(\gamma_0-\gamma_3)+f_4\gamma_2(\gamma_0-\gamma_3)\Big)\chi,\\
&\quad \text{where } f_i=f_i[x_2(x_0-x_3)^{-1}] \text{ are determined by (2.4.61),}\\
&\quad \text{(2.4.62) with } \theta=(2k-1)/2;\\
\psi(x) &= (x_0-x_3)^{-k}\exp\{(1/2)(\gamma_0-\gamma_3)[\gamma_1x_1(x_0-x_3)-\gamma_2\\
&\quad \times\ln(x_0-x_3)]\}\Big(f_1+f_2\gamma_2+f_3(\gamma_0-\gamma_3)+f_4\gamma_2(\gamma_0-\gamma_3)\Big)\chi,\\
&\quad \text{where } f_i=f_i[\ln(x_0-x_3)+x_2(x_0-x_3)^{-1}-1] \text{ are determined}\\
&\quad \text{by (2.4.61), (2.4.62) with } \theta=(2k-1)/2.
\end{aligned}
$$

3. Conformally-invariant solutions of the nonlinear Dirac-Gürsey equation. Substitution of the $C(1,3)$-invariant Ansätze for spinor field listed in (2.2.29) into the Dirac-Gürsey equation yields systems of ODEs (2.3.23) with $R=\lambda(\bar\varphi\varphi)^{1/3}$.

In spite of the extremely complicated structure of equations (2.3.23) some of them can be integrated in quadratures within the framework of the above described approach.

Lemma 2.4.3. *The quantities*

$$
\begin{aligned}
&I_3=\bar\varphi\varphi,\quad I_4=\bar\varphi\varphi\exp\{4\omega\},\\
&I_8=\bar\varphi\varphi\,\omega^{-3/2},\quad I_9=\bar\varphi\varphi,\\
&I_{10}=\bar\varphi\varphi\,\omega^{1/2}(\omega-4)^{1/2}[\omega^{1/2}+(\omega-4)^{1/2}]
\end{aligned}
$$

are the first integrals of the systems of ODEs 3, 4, 8–10 from (2.3.23).

We will prove the lemma for the system of ODEs 8, other systems are treated in the same way.

Multiplying the mentioned system by the matrix $-(i/2\omega)\gamma_2$ on the left yields

$$\dot{\varphi} = (3/4)\omega^{-1}\varphi - (i\lambda/2)\omega^{-7/6}(\bar{\varphi}\varphi)^{1/3}\gamma_2\varphi, \tag{2.4.63}$$

the conjugate spinor satisfying the equation

$$\dot{\bar{\varphi}} = (3/4)\omega^{-1}\bar{\varphi} + (i\lambda/2)\omega^{-7/6}(\bar{\varphi}\varphi)^{1/3}\bar{\varphi}\gamma_2. \tag{2.4.64}$$

Multiplying (2.4.63) by $\bar{\varphi}$ on the left, (2.4.64) by φ on the right and summing we come to the ODE for $\bar{\varphi}\varphi$

$$(\bar{\varphi}\varphi)^{\cdot} = (3/2\omega)\bar{\varphi}\varphi,$$

whence $\bar{\varphi}\varphi = C\omega^{3/2}$ or $\bar{\varphi}\varphi\,\omega^{-3/2} = C = \mathrm{const}$. The assertion is proved. $\rhd$

Applying the above lemma we can construct general solutions of nonlinear systems of ODEs 3, 4, 8–10 from (2.3.23) with the help of normalized solutions of their linearized versions. And what is more, normalized solutions of the linearized systems of ODEs 3, 8–10 can be obtained in explicit form. This fact enables us to integrate in quadratures the systems of nonlinear ODEs 3, 8–10 from (2.3.23).

$$\begin{aligned}
\varphi(\omega) &= \exp\{i\lambda(\bar{\chi}\chi)^{1/3}(\gamma_2+\gamma_3-\gamma_0)\omega\}\chi,\\
\varphi(\omega) &= \omega^{3/4}\exp\{(3i\lambda/2)(\bar{\chi}\chi)^{1/3}\gamma_2\omega^{1/3}\}\chi,\\
\varphi(\omega) &= \exp\{-i\lambda(\bar{\chi}\chi)^{1/3}\gamma_1\omega\}\chi,\\
\varphi(\omega) &= \omega^{-1/4}(\omega-4)^{-1/4}\Big(\omega^{1/2}+(\omega-4)^{1/2}\Big)^{-1/2}\\
&\quad\times\exp\Big\{-i2^{-4/3}\lambda(\bar{\chi}\chi)^{1/3}\gamma_2\int^{\omega} z^{-2/3}(z-4)^{-2/3}dz\Big\}\chi,
\end{aligned}$$

where χ is an arbitrary constant four-component column.

Substitution of the above expressions into the corresponding Ansätze for the spinor field $\psi(x)$ listed in (2.2.29) yields four classes of exact solutions of the conformally-invariant nonlinear Dirac-Gürsey equation (1.2.26)

$$\begin{aligned}
\psi(x) &= [1+(x_0-x_3)^2]^{-1}R[\arctan(x_0-x_3)]\exp\{-(1/2)\gamma_1\gamma_2\\
&\quad\times\arctan(x_0-x_3)\}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)\arctan(x_0-x_3)\}\\
&\quad\times\exp\{-(1/2)\gamma_2(\gamma_0-\gamma_3)[x_2(x_0-x_3)-x_1]
\end{aligned}$$

$$
\begin{aligned}
& \times[1+(x_0-x_3)^2]^{-1}\}\exp\Big\{i\lambda(\bar{\chi}\chi)^{1/3}(\gamma_2+\gamma_3-\gamma_0) \\
& \times\Big(-\arctan(x_0-x_3)+[x_1(x_0-x_3)+x_2][1+(x_0-x_3)^2]^{-1}\Big)\Big\}\chi, \\
\psi(x) \;=\; & (x_1^2+x_2^2)^{-1/4}[1+(x_0-x_3)^2]^{-3/4}R[\arctan(x_0-x_3)] \\
& \times\exp\{-(1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\}\exp\{(3i\lambda/2)(\bar{\chi}\chi)^{1/3}\gamma_2 \\
& \times(x_1^2+x_2^2)^{1/3}[1+(x_0-x_3)^2]^{-1/3}\}\chi, \\
\psi(x) \;=\; & [1+(x_0-x_3)^2]^{-1}R[\arctan(x_0-x_3)]\exp\{-(1/2)\gamma_1\gamma_2 \\
& \times\arctan(x_0-x_3)\}\exp\{-(1/2)\gamma_1(\gamma_0-\gamma_3)[x_1(x_0-x_3)+x_2] \\
& \times[1+(x_0-x_3)^2]^{-1}\}\exp\{-i\lambda(\bar{\chi}\chi)^{1/3}\gamma_1[x_2(x_0-x_3)-x_1] \\
& \times[1+(x_0-x_3)^2]^{-1}\}\chi, \\
\psi(x) \;=\; & (x_1^2+x_2^2)^{-1}\{\cos(\tau_2/2)\cos(\tau_3/2)+\gamma_0\gamma_3\sin(\tau_2/2)\sin(\tau_3/2) \\
& +\gamma\cdot x[\gamma_0\sin(\tau_2/2)\cos(\tau_3/2)-\gamma_3\cos(\tau_2/2)\sin(\tau_3/2)]\} \\
& \times\exp\{-(1/2)\gamma_1\gamma_2\arctan(x_1/x_2)\}\omega^{-1/4}(\omega-4)^{-1/4} \\
& \times[\omega^{1/2}+(\omega-4)^{1/2}]^{-1/2}\exp\Big\{-i2^{-4/3}\lambda(\bar{\chi}\chi)^{1/3}\gamma_2\int^{\omega}z^{-2/3} \\
& \times(z-4)^{-2/3}dz\Big\}\chi,
\end{aligned}
$$

where the following notations are used

$$
\begin{aligned}
& R(\tau)=\cos^2(\tau/2)+\gamma_0\gamma_3\sin^2(\tau/2)+(1/2)\gamma\cdot x(\gamma_0-\gamma_3)\sin\tau, \\
& \tau_2=\arctan[(x\cdot x-1)(2x_0)^{-1}]+\pi/2, \\
& \tau_3=\arctan[(x\cdot x+1)(2x_3)^{-1}]-\pi/2, \\
& \omega=[4x_0^2+(x\cdot x-1)^2](x_1^2+x_2^2)^{-1}.
\end{aligned}
$$

4. Exact solutions of equation (2.4.2). To construct exact solutions of system of nonlinear PDEs (2.4.2) we use $P(1,3)$-invariant Ansätze for the spinor field (2.2.8) and Ansätze (2.3.16). Omitting intermediate computations we give the $P(1,3)$-ungenerable families of exact solutions of the nonlinear spinor equation (2.4.2) (see, also, [135, 137]):

$$
\begin{aligned}
\psi_1(x) \;=\; & \exp\{-i\theta(\gamma\cdot a)(a\cdot x)\}\chi, \\
\psi_2(x) \;=\; & \exp\{i\theta(\gamma\cdot b)(b\cdot x)\}\chi, \\
\psi_3(x) \;=\; & \exp\{(1/2)(\gamma\cdot a)(\gamma\cdot d)\ln(a\cdot z+d\cdot z)\} \\
& \times\exp\{i\gamma\cdot c[\theta-(i/2)(\gamma\cdot a+\gamma\cdot d)]c\cdot z\}\chi,
\end{aligned}
$$

$$\begin{aligned}
\psi_4(x) &= \exp\{-(1/2)(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\}\\
&\quad\times\exp\{(i\theta/2)(\gamma\cdot b)[2b\cdot z+(a\cdot z+d\cdot z)^2]\}\chi,\\
\psi_5(x) &= \exp\{-(1/2)(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\}\\
&\quad\times\exp\{(i\theta/2)(1+\alpha^2)^{-1}(\gamma\cdot c-\alpha\gamma\cdot b)\\
&\quad\times[2(c\cdot z-\alpha b\cdot z)-\alpha(a\cdot z+d\cdot z)^2]\}\chi,\\
\psi_6(x) &= \exp\{(1/2)[b\cdot z-\ln(a\cdot z+d\cdot z)](a\cdot z+d\cdot z)^{-1}\\
&\quad\times(\gamma\cdot a+\gamma\cdot d)\gamma\cdot b\}\exp\{(1/2)(\gamma\cdot a)(\gamma\cdot d)\ln(a\cdot z+d\cdot z)\}\\
&\quad\times\exp\Big\{\Big(\gamma\cdot c(\gamma\cdot a+\gamma\cdot d)+i\theta[\gamma\cdot c-\beta(\gamma\cdot a+\gamma\cdot d)]\Big)\\
&\quad\times(c\cdot z-\beta\ln[a\cdot z+d\cdot z])\Big\}\chi,\\
\psi_7(x) &= \exp\{[-(1/2)(\dot w_1\gamma\cdot b+\dot w_2\gamma\cdot c)+w_3\gamma_4](\gamma\cdot a+\gamma\cdot d)\}\\
&\quad\times\exp\{i\theta\gamma\cdot b(b\cdot z+w_1)\}\chi,\\
\psi_8(x) &= [(b\cdot z+w_1)^2+(c\cdot z+w_2)^2]^{-1/4}\\
&\quad\times\exp\{(-(1/2)[\dot w_1\gamma\cdot b+\dot w_2\gamma\cdot c)+w_3\gamma_4](\gamma\cdot a+\gamma\cdot d)\}\\
&\quad\times\exp\{-(1/2)(\gamma\cdot b)(\gamma\cdot c)\arctan[(b\cdot z+w_1)/(c\cdot z+w_2)]\}\\
&\quad\times\exp\{i\gamma\cdot cf[(b\cdot z+w_1)^2+(c\cdot z+w_2)^2]\}\chi.
\end{aligned}$$

Here we use the following notations:

$$f(\omega)=\begin{cases} m\omega^{1/2}+\lambda(1-k)^{-1}(\bar\chi\chi)^k\omega^{(1-k)/2}, & k\neq 1,\\ m\omega^{1/2}+(\lambda/2)(\bar\chi\chi)\ln\omega, & k=1;\end{cases}$$

$z_\mu=x_\mu+\theta_\mu$; $\theta=m+\lambda(\bar\chi\chi)^k$; $w_a=w_a(d\cdot z+d\cdot z)$ are arbitrary smooth real-valued functions; $\alpha,\ \beta,\ \theta_\mu$ are real constants.

As earlier, we denote by $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ arbitrary real parameters satisfying (2.4.47).

2.5. Nonlinear spinor equations and special functions

Here we will establish a rather unexpected fact: there exists a correspondence between exact solutions of the nonlinear Dirac equation

$$\{i\gamma_\mu\,\partial_\mu-F(\bar\psi\psi)\}\psi(x)=0, \tag{2.5.1}$$

where $F\in C^1(\mathbb{R}^1,\mathbb{R}^1)$, and special functions satisfying a second-order ODE of the form

$$\ddot U+a_1(\omega)\dot U+a_2(\omega)U=0. \tag{2.5.2}$$

The above facts enable us to construct exact solutions of equation (2.5.1) in terms of the Weierstrass, Gauss and Chebyshev-Hermite functions.

To obtain exact solutions of PDE (2.5.1) we use the following Ansätze:

$$\begin{aligned}\psi(x) &= \exp\{(1/2)x_1(x_0+x_3)^{-1}(\gamma_0+\gamma_3)\gamma_1\} \\ &\quad\times \exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\varphi\,(x_0^2-x_1^2-x_3^2),\end{aligned} \tag{2.5.3}$$

$$\begin{aligned}\psi(x) &= \exp\{(1/2)\,(x_0+x_3)^{-1}(\gamma_0+\gamma_3)(\gamma_1 x_1+\gamma_2 x_2)\} \\ &\quad\times \exp\{(1/2)\gamma_0\gamma_3\ln(x_0+x_3)\}\varphi\,(x\cdot x).\end{aligned} \tag{2.5.4}$$

Substituting (2.5.3), (2.5.4) into the initial equation (2.5.1) we get systems of ODEs for the four-component functions $\varphi = \varphi\,(\omega)$

$$4\omega\dot{\varphi} = -\Big\{n(1+\gamma_0\gamma_3) + iF(\bar{\varphi}\varphi)\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\Big\}\varphi, \tag{2.5.5}$$

where the cases $n = 2$ and $n = 3$ correspond to Ansätze (2.5.3) and (2.5.4) accordingly.

The equation for the conjugate spinor $\bar{\varphi}$ has the form

$$4\omega\dot{\bar{\varphi}} = -\bar{\varphi}\Big\{n(1-\gamma_0\gamma_3) - iF(\bar{\varphi}\varphi)\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\Big\}. \tag{2.5.6}$$

Multiplying equation (2.5.5) by $\bar{\varphi}$, equation (2.5.6) by φ and summing yield the ODE for a scalar function $\bar{\varphi}\varphi$

$$4\omega(\bar{\varphi}\varphi)^{\cdot} = -2n\bar{\varphi}\varphi,$$

which general solution reads

$$\bar{\varphi}\varphi = C\omega^{-n/2}, \quad C = \text{const}. \tag{2.5.7}$$

Thus, equation (2.5.5) is reduced to the linear ODE

$$4\omega\dot{\varphi} = -\Big\{n(1+\gamma_0\gamma_3) + iF(C\omega^{-n/2})\Big(\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\Big\}\varphi \tag{2.5.8}$$

with the nonlinear additional condition (2.5.7).

If we choose γ-matrices in the representation (1.1.8), then equation (2.5.8) in component-wise notation takes the form

$$\begin{aligned}&2\dot{\varphi}^0 = -iF(C\omega^{-n/2})\varphi^2, \quad 2\omega\dot{\varphi}^1 = -iF(C\omega^{-n/2})\varphi^3 - n\varphi^1, \\ &2\dot{\varphi}^3 = -iF(C\omega^{-n/2})\varphi^1, \quad 2\omega\dot{\varphi}^2 = -iF(C\omega^{-n/2})\varphi^0 - n\varphi^2.\end{aligned}$$

On making the change of the independent variable

$$t = C\omega^{-n/2}, \quad \omega = (t/C)^{-2/n}, \tag{2.5.9}$$

we get

$$\begin{aligned} &nC^{-2/n}t^{(n+2)/n}\varphi^0_t = iF(t)\varphi^2, \quad nt\varphi^1_t = iF(t)\varphi^3 + n\varphi^1,\\ &nC^{-2/n}t^{(n+2)/n}\varphi^3_t = iF(t)\varphi^1, \quad nt\varphi^2_t = iF(t)\varphi^0 + n\varphi^2. \end{aligned} \tag{2.5.10}$$

System of ODEs (2.5.10) by means of the change of the independent variable

$$\xi = \int\limits_a^t F(\tau)\tau^{-n/2}d\tau \tag{2.5.11}$$

is reduced to the form

$$\begin{aligned} &nC^{-2/n}\varphi^0_\xi = it^{-1}\varphi^2, \quad nt^{(n-2)/n}\varphi^1_\xi = i\varphi^3 + nF^{-1}(t)\varphi^1,\\ &nC^{-2/n}\varphi^3_\xi = it^{-1}\varphi^1, \quad nt^{(n-2)/n}\varphi^2_\xi = i\varphi^0 + nF^{-1}(t)\varphi^2. \end{aligned} \tag{2.5.12}$$

Differentiating the first equation with respect to ξ we get a second-order ODE of the form

$$R_{\xi\xi} + C^{2/n}n^{-2}t^{2(1-n)/n}R = 0, \tag{2.5.13}$$

where the function $t = t(\xi)$ is determined by (2.5.11).

Consequently, system (2.5.12) is equivalent to the following second-order system of ODEs:

$$\begin{aligned} &\varphi^0_{\xi\xi} + C^{2/n}n^{-2}\Big(t(\xi)\Big)^{2(1-n)/n}\varphi^0 = 0, \quad \varphi^1 = -in\,t(\xi)C^{-2/n}\varphi^3_\xi,\\ &\varphi^3_{\xi\xi} + C^{2/n}n^{-2}\Big(t(\xi)\Big)^{2(1-n)/n}\varphi^3 = 0, \quad \varphi^2 = -in\,t(\xi)C^{-2/n}\varphi^0_\xi. \end{aligned} \tag{2.5.14}$$

Let $u(\xi)$, $v(\xi)$ be a fundamental system of solutions of equation (2.5.13). Then, the general solution of system (2.5.14) is represented in the form

$$\begin{aligned} \varphi^0 &= \chi^0u(\xi) + \chi^2v(\xi),\\ \varphi^1 &= -in\,t(\xi)C^{-2/n}\Big(\chi^3\dot{u}(\xi) + \chi^1\dot{v}(\xi)\Big),\\ \varphi^2 &= -in\,t(\xi)C^{-2/n}\Big(\chi^0\dot{u}(\xi) + \chi^2\dot{v}(\xi)\Big),\\ \varphi^3 &= \chi^3u(\xi) + \chi^1v(\xi), \end{aligned} \tag{2.5.15}$$

where $\chi^0,\ \chi^1,\ \chi^2,\ \chi^3$ are arbitrary complex constants.

Formulae (2.5.15) give the general solution of system of nonlinear ODEs (2.5.5) if equality (2.5.7) holds. Substitution of (2.5.15) into (2.5.7) gives rise to the following relation for $C,\ \chi^\mu$:

$$\begin{aligned}\bar{\psi}\psi &= \psi^{0*}\psi^2 + \psi^{2*}\psi^0 + \psi^{1*}\psi^3 + \psi^{3*}\psi^1 = intC^{-2/n}\\ &\quad \times(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*} - \chi^1\chi^{3*})w(u,v) = t = C\omega^{-n/2}.\end{aligned}$$

Here $w(u,v) = u\dot{v} - \dot{u}v$ is the Wronskian of the fundamental system of solutions of equation (2.5.13) which is constant for any $u,\ v$ satisfying (2.5.13).

The above relation is rewritten in the form

$$C = \{in(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*} - \chi^1\chi^{3*})w(u,v)\}^{n/2}. \tag{2.5.16}$$

It is well-known that each ODE of the form (2.5.2) is transformed to equation (2.5.13) by an appropriate change of variables (one has to take into account that the function $t = t(\xi)$ depends on arbitrary function F). Consequently, choosing the function $F(\bar{\varphi}\varphi)$ in an appropriate way we can obtain exact solutions of the nonlinear Dirac equation in terms of any special function described by equation (2.5.2).

We will consider several particular cases of equation (2.5.13). First of all, we recall that solutions of equation (2.5.13) (and, consequently, solutions of the nonlinear Dirac equation (2.5.1) of the form (2.5.3), (2.5.4)) under $F = \lambda(\bar{\varphi}\varphi)^{1/2k}$ are expressed in terms of the Bessel functions (see Section 2.4).

1. Choosing

$$n^{-2}C^{2/n}t^{2(1-n)/n} = 2N + 1 - \xi^2, \quad N \in \mathbb{N}, \tag{2.5.17}$$

in (2.5.13) yields the Weber equation

$$\ddot{R} + (2N + 1 - \xi^2)R = 0.$$

The fundamental system of solutions of the above equation reads [197]

$$\begin{aligned} u(\xi) &= \exp\Big\{-(1/2)\xi^2\Big\}H_N(\xi),\\ v(\xi) &= u(\xi)\int\limits_a^\xi \Big(u(\tau)\Big)^{-2}d\tau, \quad a = \text{const},\end{aligned} \tag{2.5.18}$$

where

$$H_N(\xi) = (-1)^N \exp\{\xi^2\}\frac{d^N}{d\xi^N}\exp\{-\xi^2\}$$

is the Chebyshev-Hermite polynomial.

It is not difficult to verify that functions (2.5.18) satisfy the identity

$$w(u,v) = 1.$$

Thus, substitution of formulae (2.5.15), (2.5.18) into (2.5.3), (2.5.4) with account of (2.5.16) under $w(u,v) = 1$ gives rise to a class of the exact solutions of the nonlinear Dirac equation in terms of the Chebyshev-Hermite polynomials and what is more

$$\xi^2 = 2N + 1 - n^{-2}C^{(4-2n)/n}\omega^{n-1}. \tag{2.5.19}$$

To obtain an explicit form of $F = F(t)$ we differentiate equality (2.5.17) with respect to t

$$2(1-n)n^{-3}C^{2/n}t^{(2-3n)/n} = -2\xi\frac{d\xi}{dt},$$

whence it follows that

$$F(t) = (n-1)n^{-3}C^{2/n}t^{(4-3n)/n}\Big(2N - n^{-2}C^{2/n}t^{2(1-n)/n}\Big)^{-1/2}.$$

Let us note that under $n = 3$, $i(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*} - \chi^1\chi^{3*}) < 0$ the solution obtained is localized in the Minkowski space with exception of the hyperplane $x_3 = -x_0$, where it has a non-integrable singularity.

2. If we choose

$$n^{-2}C^{2/n}t^{2(1-n)/n} = -(3/4)\mathrm{We}(\xi), \tag{2.5.20}$$

where $\mathrm{We}(\xi)$ is the Weierstrass function having the invariants ω_1, ω_2, in (2.5.13), then the Lamé equation is obtained

$$\ddot{R} - (3/4)\mathrm{We}(\xi)R = 0. \tag{2.5.21}$$

The fundamental system of solutions of ODE (2.5.21) is as follows [197]

$$\begin{aligned} u(\xi) &= \{\dot{\mathrm{W}}\mathrm{e}(\xi/2)\}^{-1/2}, \\ v(\xi) &= \mathrm{We}(\xi/2)\{\dot{\mathrm{W}}\mathrm{e}(\xi/2)\}^{-1/2} \end{aligned} \tag{2.5.22}$$

and what is more $w(u,v) = 1/2$. Hence, using formulae (2.5.3), (2.5.4), (2.5.15), (2.5.16) (under $w(u,v) = 1/2$) we obtain the exact solutions of the

initial PDE (2.5.1) in terms of the Weierstrass function, the equalities

$$\xi = \int\limits_{-\infty}^{(4/3)n^{-2}C^{(4-2n)/n}\omega^{n-1}} (-4\tau^3 + \omega_1\tau - \omega_2)^{-1/2}d\tau,$$

$$F(t) = (4/3)(n-1)n^{-2}C^{2/n}t^{(2-3n)/n}\Big\{-4\Big((4/3)n^{-2}C^{2/n} \times t^{2(1-n)/n}\Big)^3 + (4/3)\omega_1 n^{-2}C^{2/n}t^{2(1-n)/n} - \omega_2\Big\}^{-1/2}$$

holding.

3. Choosing in (2.5.13)

$$n^{-2}C^{2/n}t^{2(1-n)/n} = (1/4)\xi^{-2}\{2(a+b+1) - (a+b+1)^2\} - ab$$

we get the hypergeometric equation

$$\ddot{R} + \Big((1/4)\xi^{-2}[1-(a+b)^2] - ab\Big)R = 0.$$

The fundamental system of solutions of this equation is as follows [197]

$$\begin{aligned} u(\xi) &= \xi^{(1+a+b)/2}F(a,\, b,\, a+b+1,\, \xi), \\ v(\xi) &= \xi^{(1-a-b)/2}F(-b,\, -a\,, 1-a-b,\, \xi), \end{aligned} \tag{2.5.23}$$

where $F = F(a,\, b\,, c,\, \xi)$ is the hypergeometric Gauss function and besides

$$\begin{aligned} w(u,v) = (a+b)\Gamma(1+a+b)\Gamma(1-a-b) \\ \times\Big\{\Gamma(1+a)\Gamma(1+b)\Gamma(1-a)\Gamma(1-b)\Big\}^{-1}. \end{aligned} \tag{2.5.24}$$

Here $\Gamma = \Gamma(a)$ is the Euler γ-function.

Substitution of formulae (2.5.15), (2.5.16), (2.5.23), (2.5.24) into the Ansätze (2.5.3), (2.5.4) yields the exact solutions of the nonlinear Dirac equation (2.5.1), the relations

$$\xi = (1/2)\Big(1-(a+b)^2\Big)^{1/2}\Big(n^{-2}C^{(4-2n)/n}\omega^{n-1} + ab\Big)^{-1/2},$$

$$F(t) = (1/2)n^{-3}(n-1)C^{2/n}\Big(1-(a+b)^2\Big)^{1/2}t^{(4-3n)/n} \times \Big(n^{-2}C^{2/n}t^{2(1-n)/n} + ab\Big)^{-3/2}$$

holding.

Let us note that solutions of equation (2.5.1) of the form (2.5.3), (2.5.4) can be treated as solutions of the linear Dirac equation

$$\{i\gamma_\mu \partial_\mu - U(x)\}\psi(x) = 0 \tag{2.5.25}$$

with potentials $U(x) = F[C(x_0^2 - x_1^2 - x_3^2)^{-1}]$ and $U(x) = F[C(x \cdot x)^{-3/2}]$. That is why there exists an analogy between equations (2.5.1) and (2.5.25). The principal difference is that in the case of a linear equation the potential characterizes interaction of the spinor field with some external field (for example, with the scalar field $u(x) = U(x)$), while in the nonlinear case the "potential" is determined by self-interaction of the spinor field $\psi(x)$.

2.6. Construction of fields with spins $s = 0, 1, 3/2$ via the Dirac field

In [152] we have suggested a purely algebraic method of construction of Ansätze for scalar, vector and tensor fields by the use of Ansätze for the spinor field $\psi(x)$. The method is based on the following well-known fact: provided the spinor field $\psi(x)$ transforms according to formulae (1.1.24)–(1.1.26), then the quantities

$$u(x) = \bar{\psi}\psi, \tag{2.6.1}$$

$$A_\mu(x) = \bar{\psi}\gamma_\mu\psi, \tag{2.6.2}$$

$$F_{\mu\nu}(x) = i\bar{\psi}\gamma_\mu\gamma_\nu\psi, \tag{2.6.3}$$

transform with respect to the Poincaré group as the scalar, vector and second-rank tensor correspondingly. Consequently, substitution of the $P(1,3)$-invariant Ansätze for $\psi(x)$ obtained in Section 2.2 into formulae (2.6.1)–(2.6.3) with subsequent replacement $\bar{\varphi}\varphi \to B(\omega)$, $\bar{\varphi}\gamma_\mu\varphi \to B_\mu(\omega)$, $i\bar{\varphi}\gamma_\mu\gamma_\nu\varphi \to B_{\mu\nu}(\omega)$ yields the Ansätze for the scalar, vector and tensor fields invariant under the one- and three-dimensional subalgebras of the algebra $AP(1,3)$.

It is worth noting that the above described procedure of construction of invariant Ansätze is much simpler than integration of system of PDEs (1.5.22), (1.5.20).

Furthermore, if we substitute Ansätze for $\psi(x)$ invariant under one- and three-dimensional subalgebras of the Lie algebra of the extended Poincaré group $A\widetilde{P}(1,3)$ into formulae (2.6.1)–(2.6.3), then $\widetilde{P}(1,3)$-invariant Ansätze for fields $u(x)$, $A_\mu(x)$, $F_{\mu\nu}(x)$ are obtained.

To construct conformally-invariant Ansätze for the scalar, vector and tensor fields we introduce into formulae (2.6.1)–(2.6.3) the normalizing factors of the form $(\psi\bar{\psi})^\alpha$

$$u(x) = (\bar{\psi}\psi)^{1/3}, \tag{2.6.4}$$

$$A_\mu(x) = \bar{\psi}\gamma_\mu\psi(\bar{\psi}\psi)^{-2/3}, \tag{2.6.5}$$

$$F_{\mu\nu}(x) = i\bar{\psi}\gamma_\mu\gamma_\nu\psi(\bar{\psi}\psi)^{-1/3} \tag{2.6.6}$$

(it is not difficult to ascertain that the fields $u(x)$, $A_\mu(x)$ transform according to formulae (1.4.5), (1.4.13) provided the spinor field $\psi(x)$ transforms according to (1.1.28)).

We apply the procedure described to obtain Poincaré-invariant Ansätze for the vector field $A_\mu(x)$ which reduce the corresponding $P(1,3)$-invariant system of PDEs to ODEs. Before substituting Ansätze for $\psi(x)$ into formula (2.6.2) we generate them by transformations from the Poincaré group (formulae (2.4.41)–(2.4.43)). Substitution of $P(1,3)$-ungenerable Ansätze for the spinor field $\psi(x)$ into (2.6.2) yields $P(1,3)$-ungenerable Ansätze for the vector field $A_\mu(x)$ that can be represented in the following unified form:

$$\begin{aligned} A_\mu(x) \;=\; & \Big\{(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 \\ & +2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 \\ & -\theta_1\sin\theta_3)c_\nu + (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu \\ & -b_\nu c_\mu)\sin\theta_3 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0} \\ & \times(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu)\Big\}B^\nu(\omega), \end{aligned} \tag{2.6.7}$$

where $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ are arbitrary real constants satisfying equalities (2.4.47) and B^ν are arbitrary smooth functions. Explicit forms of the functions $\theta_\mu,\ \omega$ depend on the choice of a three-dimensional subalgebra of the Poincaré algebra (2.2.7) and are given below

1) $\theta_\mu = 0, \quad \omega = d\cdot z;$
2) $\theta_\mu = 0, \quad \omega = a\cdot z;$
3) $\theta_\mu = 0, \quad \omega = k\cdot z;$
4) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = (a\cdot z)^2 - (d\cdot z)^2;$
5) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = b\cdot z;$
6) $\theta_0 = -\alpha^{-1}(c\cdot z), \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = b\cdot z, \quad \alpha \neq 0;$
7) $\theta_0 = -\alpha^{-1}(c\cdot z), \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = \alpha\ln(k\cdot z) - c\cdot z, \quad \alpha \neq 0;$

8) $\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(b\cdot z/c\cdot z), \quad \omega = (b\cdot z)^2 + (c\cdot z)^2;$
9) $\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\alpha^{-1}(a\cdot z), \quad \omega = d\cdot z, \quad \alpha \neq 0;$
10) $\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha^{-1}(d\cdot z), \quad \omega = a\cdot z, \quad \alpha \neq 0;$
11) $\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = (d\cdot z - a\cdot z)/2, \quad \omega = k\cdot z;$
12) $\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = b\cdot z/2k\cdot z, \quad \omega = k\cdot z;$
13) $\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = (\alpha b\cdot z - c\cdot z)(2\alpha k\cdot z)^{-1}, \quad \omega = k\cdot z,$
$\alpha \neq 0;$
14) $\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = (c\cdot z)/2, \quad \omega = k\cdot z;$ (2.6.8)
15) $\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -(k\cdot z)/2, \quad \omega = 2b\cdot z + (k\cdot z)^2;$
16) $\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -(k\cdot z)/2, \quad \omega = 2(c\cdot z - \alpha b\cdot z) - \alpha(k\cdot z)^2;$
17) $\theta_0 = \alpha^{-1}\arctan(b\cdot z/c\cdot z), \quad \theta_1 = \theta_2 = 0,$
$\theta_3 = -\arctan(b\cdot z/c\cdot z), \quad \omega = (b\cdot z)^2 + (c\cdot z)^2, \quad \alpha \neq 0;$
18) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha\ln(k\cdot z),$
$\omega = (a\cdot z)^2 - (d\cdot z)^2;$
19) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(b\cdot z/c\cdot z),$
$\omega = (b\cdot z)^2 + (c\cdot z)^2;$
20) $\theta_0 = \theta_3 = 0, \quad \theta_1 = b\cdot z/2k\cdot z, \quad \theta_2 = c\cdot z/2k\cdot z, \quad \omega = k\cdot z;$
21) $\theta_0 = \theta_3 = 0, \quad \theta_1 = (1/2)[(k\cdot z + \beta)b\cdot z - \alpha c\cdot z][k\cdot z(k\cdot z + \beta)$
$-\alpha]^{-1}, \quad \theta_2 = (1/2)(k\cdot zc\cdot z - b\cdot z)[k\cdot z(k\cdot z + \beta) - \alpha]^{-1},$
$\omega = k\cdot z;$
22) $\theta_0 = \theta_3 = 0, \quad \theta_1 = (1/2k\cdot z)(b\cdot z - c\cdot z(k\cdot z + \beta)^{-1}),$
$\theta_2 = (1/2)c\cdot z(k\cdot z + \beta)^{-1}, \quad \omega = k\cdot z;$
23) $\theta_0 = \theta_3 = 0, \quad \theta_1 = b\cdot z/2k\cdot z, \quad \theta_2 = (1/2)c\cdot z(k\cdot z + 1)^{-1},$
$\omega = k\cdot z;$
24) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = b\cdot z/2k\cdot z,$
$\theta_2 = \theta_3 = 0, \quad \omega = (a\cdot z)^2 - (b\cdot z)^2 - (d\cdot z)^2;$
25) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = [b\cdot z - \alpha\ln(k\cdot z)]/2k\cdot z,$
$\theta_2 = \theta_3 = 0, \quad \omega = c\cdot z - \beta\ln(k\cdot z);$
26) $\theta_0 = 0, \quad \theta_1 = b\cdot z/2k\cdot z, \quad \theta_2 = c\cdot z/2k\cdot z,$
$\theta_3 = -z\cdot z/4k\cdot z, \quad \omega = k\cdot z;$
27) $\theta_0 = -\ln(k\cdot z), \quad \theta_1 = b\cdot z/2k\cdot z, \quad \theta_2 = c\cdot z/2k\cdot z,$
$\theta_3 = \alpha\ln(k\cdot z), \quad \omega = z\cdot z.$

where $k_\mu = a_\mu + b_\mu,\ z_\mu = x_\mu + \tau_\mu,\ \tau_\mu = \text{const},\ \mu = 0, \dots, 3.$

Let us consider an example of construction of an Ansatz for the vector field $A_\mu(x)$ by taking as $\psi(x)$ the Ansatz

$$\psi(x) = \exp\{(1/2)\gamma_0\gamma_3 \ln(x_0 + x_3)\}\varphi\,(x_0^2 - x_3^2)$$

invariant under the three-dimensional algebra $\langle J_{03},\ P_1,\ P_2\rangle \in AP(1,3)$.

It is not difficult to check that a $P(1,3)$-ungenerable Ansatz for the spinor field is obtained by making the following change:

$$\begin{array}{llll} \gamma_0 \to \gamma\cdot a, & \gamma_1 \to \gamma\cdot b, & \gamma_2 \to \gamma\cdot c, & \gamma_3 \to \gamma\cdot d, \\ x_0 \to a\cdot z, & x_1 \to b\cdot z, & x_2 \to c\cdot z, & x_3 \to d\cdot z \end{array}$$

in the above Ansatz.

As a result, we have

$$\begin{aligned} \psi(x) &= \exp\{(1/2)\gamma\cdot a\gamma\cdot d\ln(k\cdot z)\}\varphi \\ &\quad = \{\cosh[(1/2)\ln(k\cdot z)] + \gamma\cdot a\gamma\cdot d\sinh[(1/2)\ln(k\cdot z)]\}\varphi, \\ \bar\psi(x) &= \bar\varphi\exp\{-(1/2)\gamma\cdot a\gamma\cdot d\ln(k\cdot z)\} \\ &\quad = \{\cosh[(1/2)\ln(k\cdot z)] - \gamma\cdot a\gamma\cdot d\sinh[(1/2)\ln(k\cdot z)]\}, \end{aligned}$$

where φ is an arbitrary complex-valued four-component function of $(a\cdot z)^2 - (d\cdot z)^2$. Substitution of the formulae obtained into (2.6.2) yields

$$\begin{aligned} A_\mu(x) &= \bar\varphi\{\cosh\theta - \gamma\cdot a\gamma\cdot d\sinh\theta\}\gamma_\mu\{\cosh\theta + \gamma\cdot a\gamma\cdot d\sinh\theta\}\varphi \\ &= \bar\varphi\gamma_\mu\varphi - (\sinh\theta)\bar\varphi[\gamma\cdot a\gamma\cdot d,\ \gamma_\mu](\cosh\theta + \gamma\cdot a\gamma\cdot d\sinh\theta)\varphi \\ &= \bar\varphi\gamma_\mu\varphi + 2a_\mu\bar\varphi(\gamma\cdot d\cosh\theta + \gamma\cdot a\sinh\theta)\varphi\sinh\theta - 2d_\mu \\ &\quad \times\bar\varphi(\gamma\cdot a\cosh\theta + \gamma\cdot d\sinh\theta)\varphi\sinh\theta = \{(a_\mu a_\nu - d_\mu d_\nu) \\ &\quad \times\cosh 2\theta + (a_\mu d_\nu - d_\mu a_\nu)\sinh 2\theta - b_\mu b_\nu - c_\mu c_\nu\}\bar\varphi\gamma^\nu\varphi, \end{aligned}$$

where $\theta = (1/2)\ln(k\cdot z)$. Designating the real-valued functions $\bar\varphi\gamma_\mu\varphi$ by $B_\mu,\ \mu = 0, \dots, 3$ we arrive at the Ansatz 4 from (2.6.7).

To obtain from (2.6.7) Ansätze for the vector field invariant under the three-dimensional subalgebras of the Poincarè algebra (2.2.7) we put

$$\tau_\mu = 0, \quad a_\mu = \delta_{\mu 0}, \quad b_\mu = -\delta_{\mu 1}, \quad c_\mu = -\delta_{\mu 2}, \quad d_\mu = -\delta_{\mu 3}.$$

Let us emphasize that the above procedure of construction of Ansätze for the vector, scalar and tensor fields is based on transformational properties of

the spinor field with respect to the Poincarè group only and the explicit form of the function $\psi(x)$ is not used. There arises a natural question: what equations are satisfied by functions $u(x)$, $A_\mu(x)$, $F_{\mu\nu}(x)$ defined by formulae (2.6.1)–(2.6.3) provided $\psi = \psi(x)$ is a solution of the nonlinear Dirac equation? In other words, is it possible to construct exact solutions of equations describing fields with spins $s = 0, 1, 3/2, \ldots$ with the help of exact solutions of a nonlinear PDE for the field with the spin $s = 1/2$?

It occurs that for some classes of fields the answer to this question is positive [155].

We look for solutions of the complex nonlinear d'Alembert equation

$$\partial_\mu \partial^\mu u = \lambda_1 |u|^{k_1} u, \tag{2.6.9}$$

where $\lambda_1,\ k_1$ are constants, in the form

$$u(x) = \bar{\psi}\psi\, e^{i\theta(x)}. \tag{2.6.10}$$

Here $\psi = \psi(x)$ is a solution of nonlinear spinor equation (2.4.1) and $\theta(x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$ is a phase of the field $u(x)$. With the use of exact solutions of the nonlinear Dirac equation listed in Section 2.4 we have obtained a number of exact solutions of the nonlinear d'Alembert equation which are adduced in the Table 2.6.1.

Thus, spinors $\psi = \psi(x)$ satisfying nonlinear PDE (2.4.1) give rise to complex scalar fields $u = u(x)$ which are described by the nonlinear d'Alembert equation (2.6.9). It is interesting to note that the inverse procedure is also possible. Namely, starting from a special subclass of exact solutions of the nonlinear d'Alembert equation we can obtain exact solutions of the nonlinear Dirac equation (see Section 2.1).

As straightforward computation shows, the vector field constructed with the help of formula (2.6.2), where $\psi(x)$ is a solution of the nonlinear spinor equation (2.4.1), satisfies the following system of PDEs:

$$\begin{aligned} &\Big(\partial_\mu \partial^\mu + M^2(x)\Big) A_\mu(x) = j_\nu(x), \\ &\partial_\mu A_\mu(x) = 0, \end{aligned} \tag{2.6.11}$$

functions $M(x),\ j_\mu(x)$ depending on the choice of $\psi(x)$.

For example, if we take $\psi = \psi_1(x)$, then

$$\begin{aligned} &M(x) = \lambda(\bar{\chi}\chi)^{1/2k} = \text{const}, \\ &j_\mu(x) = \lambda\bar{\chi}(\cos\lambda x_0 - i\gamma_0 \sin\lambda x_0)\gamma_\mu(\cos\lambda x_0 + i\gamma_0 \sin\lambda x_0)\chi. \end{aligned}$$

Consequently, the nonlinear spinor field gives rise to a field $A_\mu(x)$ which can be interpreted as the vector field with a variable mass $M(x)$.

Table 2.6.1. **Exact solutions of the nonlinear d'Alembert equation**

N	$u(x)$	k_1
1–10	$C\exp\{i\alpha\cdot x\}$	$k_1\in\mathbb{R}^1$
11	$C(x_1^2+x_2^2)^{-1/2}\exp\{iw_0\}$	2
12	$C[(x_1+w_1)^2+(x_2+w_2)^2]^{-1/2}\exp\{iw_0\}$	2
13	$C(x_1^2+x_2^2)^{-1/2}\exp\{iw_0\}$	2
14	$C(x_0^2-x_1^2-x_3^2)^{-1}$	1
15	$C(x\cdot x)^{-3/2}$	2/3
16	$C(x_0^2-x_3^2)^{-1/2}$	2
17	$C(x_1^2+x_2^2)^{-1/2}\exp\{iw_0\}$	2
18	$C[(x_1+w_1)^2+(x_2+w_2)^2]^{-1/2}\exp\{iw_0\}$	2
19	$Cw_0^{-2}\exp\{i(x_1+w_1)\}$	0
21	$C(x_0^2-x_1^2-x_2^2)^{-2}$	1/2
22	$C(x_0^2-x_1^2-x_2^2)^{-2}$	1/2
24	$C(x_1^2+x_2^2+x_3^2)^{-2}$	1/2
26	$C(x_0^2-x_1^2-x_3^2)^{-1}$	1
27	$C(x\cdot x)^{-3/2}$	2/3
28	$C\{[x_2+\beta(x_0+x_3)]^2+[x_1+(1/2)(x_0+x_3)^2]^2\}^{-k}$	$1/k,\ k<0$
29	$C\{[x_2+\beta(x_0+x_3)]^2+[x_1+(1/2)(x_0+x_3)^2]^2\}^{-1}$	1

Here N denotes the number of the corresponding solution of the nonlinear Dirac equation, $w_0,\ w_1,\ w_2$ are arbitrary smooth functions of x_0+x_3; $C,\ \alpha_\mu,\ \beta$ are constants.

Let us adduce an example of a tensor field with the spin $s=1$ constructed with the use of an exact solution of nonlinear PDE (2.4.1). Substituting the

four-component function $\psi(x) = \exp\{-i\lambda\gamma_0 x_0\}$ into the formulae

$$E_a = i\bar{\psi}\gamma_0\gamma_a\psi, \quad H_a = (i/2)\varepsilon_{abc}\bar{\psi}\gamma_b\gamma_c\psi$$

we get the exact solution of the Maxwell equations with the current

$$j_0 = 0, \quad j_a = -2i\lambda(\chi^\dagger\gamma_a\chi)\sin 2\lambda x_0 - 2\lambda(\bar{\chi}\gamma_a\chi)\cos 2\lambda x_0.$$

In conclusion of this section we give the formula for construction of Poincaré-invariant Ansätze for the field with the spin $s = 3/2$

$$\Lambda_\mu(x) = (\bar{\psi}\gamma_\mu\psi)\psi, \quad \mu = 0, \dots, 3. \tag{2.6.12}$$

Substitution of the $P(1,3)$-invariant Ansätze (2.2.8) into (2.6.12) gives rise to Ansätze for the field $\Lambda_\mu(x)$ with the spin $s = 3/2$ reducing the corresponding Poincaré-invariant equation to systems of ODEs.

2.7. Exact solutions of the Dirac-d'Alembert equation

Few works containing exact solutions of systems of nonlinear PDEs of the form (1.4.8) [20, 21, 287] use essentially the Ansatz for the spinor field

$$\psi(x) = \{\gamma \cdot x f(x \cdot x) + ig(x \cdot x)\}\chi \tag{2.7.1}$$

suggested by Heisenberg [180]. In (2.7.1) $\{f, g\} \subset C^1(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary real-valued functions.

The scalar field $u = u(x)$ is looked for in the form

$$u(x) = \varphi(x \cdot x), \quad \varphi \in C^2(\mathbb{R}^1, \mathbb{C}^1). \tag{2.7.2}$$

Substitution of (2.7.1), (2.7.2) into (1.4.8) under

$$F = R_2(uu^*, \bar{\psi}\psi)\psi, \quad H = R_1(uu^*, \bar{\psi}\psi)u, \quad R_i \in C(\mathbb{R}^2, \mathbb{R}^1) \tag{2.7.3}$$

gives rise to a system of three ordinary differential equations for functions f, g, φ. Consequently, a reduction of PDE (1.4.8) both by the number of independent variables and by the number of dependent variables takes place. We recall that Ansätze constructed in Section 2.2 reduce Poincaré-invariant spinor equations by the number of independent variables only.

In [148] we have suggested a generalization of the Heisenberg Ansatz which made it possible to obtain broad classes of the exact solutions of the multidimensional Dirac and Dirac-d'Alembert equations.

Following [151] we look for a solution of system of PDEs (1.4.8), (2.7.3) of the form

$$\psi(x) = \{f(\omega)\gamma_\mu\partial_\mu\omega + ig(\omega)\}\chi, \quad u(x) = \varphi(\omega), \tag{2.7.4}$$

where $\{f, g\} \subset C^1(\mathbb{R}^1, \mathbb{R}^1)$, $\varphi \in C^2(\mathbb{R}^1, \mathbb{R}^1)$, $\omega = \omega(x)$ is an arbitrary smooth real-valued function.

Since

$$\begin{aligned}
&i\gamma_\mu\partial_\mu\psi(x) = \{-\dot{g}\gamma_\mu\partial_\mu\omega + if\partial_\mu\partial^\mu\omega + i\dot{f}(\partial_\mu\omega)(\partial^\mu\omega)\}\chi,\\
&\partial_\mu\partial^\mu u(x) = \ddot{\varphi}(\partial_\mu\omega)(\partial^\mu\omega) + \dot{\varphi}\partial_\mu\partial^\mu\omega,\\
&\bar{\psi}\psi = \bar{\chi}\chi\Big(g^2 + f^2(\partial_\mu\omega)(\partial^\mu\omega)\Big),
\end{aligned}$$

substitution of formulae (2.7.4) into (1.4.8) yields the system of PDEs for $f,\ g,\ \varphi,\ \omega$:

$$\begin{aligned}
&\ddot{\varphi}(\partial_\mu\omega)(\partial^\mu\omega) + \dot{\varphi}\partial_\mu\partial^\mu\omega = R_1\varphi,\\
&\dot{f}(\partial_\mu\omega)(\partial^\mu\omega) + f\partial_\mu\partial^\mu\omega = R_2 g,\\
&\dot{g} = -R_2 f,
\end{aligned} \tag{2.7.5}$$

where $R_i = R_i(\varphi\varphi^*,\, g^2 + f^2(\partial_\mu\omega)(\partial^\mu\omega)),\ i = 1, 2.$

If we resolve two last equations with respect to $(\partial_\mu\omega)(\partial^\mu\omega)$ and $\partial_\mu\partial^\mu\omega$, then the following necessary compatibility conditions arise

$$\partial_\mu\partial^\mu\omega = F_1(\omega), \quad (\partial_\mu\omega)(\partial^\mu\omega) = F_2(\omega).$$

In other words, the scalar function $\omega = \omega(x)$ has to satisfy the d'Alembert-Hamilton system (2.1.25) and besides the functions $F_1,\ F_2$ do not vanish simultaneously. Since functions $f(\omega),\ g(\omega)$ are arbitrary, we can choose them in such a way that $\omega = \omega(x)$ satisfies system of PDEs (2.1.30), equations (2.7.5) taking the form

$$\begin{aligned}
&\varepsilon\ddot{\varphi} + F(\omega)\dot{\varphi} = R_1(\varphi\varphi^*,\, g^2 + \varepsilon f^2)\varphi,\\
&\varepsilon\dot{f} + F(\omega)f = R_2(\varphi\varphi^*,\, g^2 + \varepsilon f^2)g,\\
&\dot{g} = -R_2(\varphi\varphi^*,\, g^2 + \varepsilon f^2)f,
\end{aligned} \tag{2.7.6}$$

where $F(\omega) = \varepsilon N\omega^{-1},\ N = 0, \ldots, 3,\ \varepsilon = \pm 1.$

Thus, the problem of constructing particular solutions of the multi-dimensional system of five PDEs (1.4.8) is reduced to integration of a system of three ODEs. If we succeed in integrating system (2.7.6), then substitution of the obtained results into Ansatz (2.7.4), where $\omega = \omega(x)$ is the solution of the d'Alembert-Hamilton system (2.1.30), gives rise to exact solutions of the initial system of PDEs (1.4.8), (2.7.3).

Let us note that Ansatz (2.7.4) can be interpreted as the formula for construction of the nonlinear spinor field $\psi(x)$ satisfying the Dirac-d'Alembert system with the help of the nonlinear scalar field $\omega(x)$ satisfying the nonlinear d'Alembert-Hamilton system.

It is clear that the $P(1,3)$-invariant Ansätze obtained in Section 2.2 can also be applied to reduce the Poincaré-invariant equation (1.4.8) but the resulting systems of ODEs prove to be much more complicated than system (2.7.6).

We will construct exact solutions of system of PDEs (1.4.8), (2.7.3) having the following nonlinearities:

$$\begin{aligned} R_1 &= -\{\mu_1|u|^{k_1} + \mu_2(\bar{\psi}\psi)^{k_2}\}^2, \\ R_2 &= \lambda_1|u|^{k_1} + \lambda_2(\bar{\psi}\psi)^{k_2}, \end{aligned} \tag{2.7.7}$$

where $|u|^2 = uu^*$; $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2,\ k_1,\ k_2$ are constants.

Substitution of Ansatz (2.7.4) into system of PDEs (1.4.8), (2.7.7) yields the following equations for unknown functions $f,\ g,\ \varphi$:

$$\begin{aligned} &\lambda\ddot{\varphi} + F(\omega)\dot{\varphi} = -\{\mu_1|\varphi|^{k_1} + \tilde{\mu}_2(g_2 + \lambda f^2)^{k_2}\}^2\varphi, \\ &\lambda\dot{f} + F(\omega)f = \{\lambda_1|\varphi|^{k_1} + \tilde{\lambda}_2(g^2 + \lambda f^2)^{k_2}\}g, \\ &\dot{g} = -\{\lambda_1|\varphi|^{k_1} + \tilde{\lambda}_2(g^2 + \lambda f^2)^{k_2}\}f. \end{aligned} \tag{2.7.8}$$

Here $\tilde{\mu}_2 = \mu_2(\bar{\chi}\chi)^{k_2}$, $\tilde{\lambda}_2 = \lambda_2(\bar{\chi}\chi)^{k_2}$; $F(\omega) = N\lambda\omega^{-1}$, $N = 0,1,2,3$, $\lambda = \varepsilon = \pm 1$.

We have succeeded in constructing the general solution of system (2.7.8) provided $N = 0$. Under $N \neq 0$ some particular solutions are obtained.

1) $N = 0,\ \lambda = 1$

Multiplying the second equation of system (2.7.8) by f, the third by g and summing we have $f\dot{f} + g\dot{g} = 0 \ \Rightarrow\ f^2 + g^2 = C_1^2 = \text{const}$. Due to this fact

equations for $g,\ f$ are easily integrated

$$\begin{aligned} f(\omega) &= C_1 \sin\left\{\lambda_1 \int^{\omega} |\varphi(z)|^{k_1} dz + \tilde{\lambda}_2 C_1^{2k_2}\omega + C_2\right\}, \\ g(\omega) &= C_1 \cos\left\{\lambda_1 \int^{\omega} |\varphi(z)|^{k_1} dz + \tilde{\lambda}_2 C_1^{2k_2}\omega + C_2\right\}, \end{aligned} \qquad (2.7.9)$$

where $C_2 = \text{const}$.

Substituting (2.7.9) into the first equation of system (2.7.8) we come to the following ODE for $\varphi(\omega)$:

$$\ddot{\varphi} = -\{\mu_1|\varphi|^{k_1} + \tilde{\mu}C_1^{2k_2}\}^2\varphi.$$

On representing the complex-valued function φ in the form

$$\varphi(\omega) = \rho(\omega)e^{i\theta(\omega)}, \qquad (2.7.10)$$

where $\rho(\omega)$, $\theta(\omega)$ are real-valued functions, we rewrite this ODE as follows

$$\ddot{\rho} - \rho\dot{\theta}^2 = -\{\mu_1\rho^{k_1} + \tilde{\mu}_2C_1^{2k_2}\}^2\rho, \quad 2\dot{\theta}\dot{\rho} + \ddot{\theta}\rho = 0. \qquad (2.7.11)$$

The second equation of the above system implies that $\dot{\theta} = C_3\rho^{-1/2},\ C_3 = \text{const}$. Substitution of this result into the first equation of system (2.7.11) yields the ODE for $\rho = \rho(\omega)$

$$\ddot{\rho} = C_3^2 - \mu_1^2\rho^{2k_1+1} - 2\mu_1\tilde{\mu}_2C_1^{2k_2}\rho^{k_1+1} - \tilde{\mu}_2^2C_1^{4k_2}\rho \equiv a_+(\rho),$$

whose general solution is given by the implicit formula

$$\int^{\rho(\omega)} \left(2\int a_+(z)dz + C_4\right)^{-1/2} dz = \omega. \qquad (2.7.12)$$

Thus, the general solution of system of ODEs (2.7.8) under $N = 0,\ \lambda = 1$ has the form

$$\begin{aligned} f(\omega) &= C_1 \sin\left\{\lambda_1 \int^{\omega} \rho^{k_1}(z)dz + \tilde{\lambda}_2 C_1^{2k_2}\omega + C_2\right\}, \\ g(\omega) &= C_1 \cos\left\{\lambda_1 \int^{\omega} \rho^{k_1}(z)dz + \tilde{\lambda}_2 C_1^{2k_2}\omega + C_2\right\}, \\ \varphi(\omega) &= \rho(\omega) \exp\left\{iC_3 \int^{\omega} \rho^{-1/2}(z)dz\right\}, \end{aligned}$$

where $C_1,\ C_2,\ C_3$ are constants, $\rho(\omega)$ is defined by (2.7.12).

2) $N = 2, 3,\ \lambda = 1.$

We look for particular solutions of system (2.7.8) in the form of power functions

$$f(\omega) = C\omega^{\alpha_1}, \quad g(\omega) = D\omega^{\alpha_2}, \quad \varphi(\omega) = E\omega^{\alpha_3}.$$

Substituting these expressions into (2.7.8) and equating exponents of ω yield

$$\alpha_1 = \alpha_2, \quad \alpha_1 - 1 = \alpha_2(1 + 2k_2), \quad \alpha_3 k_1 = 2\alpha_2 k_2,$$

whence $\alpha_1 = \alpha_2 = -1/2k_2,\ \alpha_3 = 1/k_1$.

Consequently,

$$f(\omega) = C\omega^{-1/2k_2}, \quad g(\omega) = D\omega^{-1/2k_2}, \quad \varphi(\omega) = E\omega^{-1/k_1}, \tag{2.7.13}$$

parameters $C,\ D,\ E$ satisfying the system of nonlinear algebraic equations

$$\begin{aligned}
&k_1^{-2}(Nk_1 - k_1 - 1) = \{\mu_1|E|^{k_1} + \tilde{\mu}_2(C^2 + D^2)^{k_2}\}^2,\\
&(2k_2)^{-1}D = \{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(C^2 + D^2)^{k_2}\}C,\\
&(2k_2)^{-1}(2Nk_2 - 1)C = \{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(C^2 + D^2)^{k_2}\}D.
\end{aligned} \tag{2.7.14}$$

From the second and the third equations we get the equality

$$D^2C^{-2} = 1 - 2Nk_2. \tag{2.7.15}$$

The first equation of system (2.7.14) and equality (2.7.15) yield the following restrictions on the choice of parameters $k_1,\ k_2$: $k_1 > (N-1)^{-1}, \quad k_2 > (2N)^{-1}$.

Therefore, relations (2.7.14) can be rewritten in the form

$$\begin{aligned}
&D = \varepsilon(2Nk_2 - 1)^{1/2}C,\\
&\{\mu_1|E|^{k_1} + \tilde{\mu}_2(2Nk_2C^2)^{k_2}\}^2 = (1 + k_1 - Nk_1)k_1^{-2},\\
&\{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(2Nk_2C^2)^{k_2}\} = \varepsilon(2Nk_2 - 1)^{1/2}(2k_2)^{-1},
\end{aligned} \tag{2.7.16}$$

where $\varepsilon = \pm 1,\ k_1 > (N-1),\ k_2 > (2N)^{-1}$.

Under $k_1 = 2(N-1)^{-1},\ k_2 = N^{-1}$ system (2.7.8) possesses the following class of particular solutions:

$$\begin{aligned}
f(\omega) &= \theta\omega(1 + \theta^2\omega^2)^{-(N+1)/2},\\
g(\omega) &= (1 + \theta^2\omega^2)^{-(N+1)/2},\\
\varphi(\omega) &= E(1 + \theta^2\omega^2)^{(1-N)/2},
\end{aligned}$$

parameters $\theta,\ E$ being determined by the system of nonlinear algebraic equations

$$\begin{aligned}\theta^2(N^2-1) &= \{\mu_1|E|^{2/(N-1)}+\tilde{\mu}_2\}^2\\ (N+1)\theta &= \{\lambda_1|E|^{2/(N-1)}+\tilde{\lambda}_2\}.\end{aligned} \tag{2.7.17}$$

3) $N=0,\ \lambda=-1$.

Multiplying the second equation of system (2.7.8) by f, the third by g and summing we have $f^2-g^2=-C_1^2=\text{const}$. Due to this fact equations for $g,\ f$ are easily integrated

$$\begin{aligned}f(\omega) &= C_1\sinh\biggl\{-\lambda_1\int^{\omega}|\varphi(z)|^{k_1}dz-\tilde{\lambda}_2C_1^{2k_2}\omega+C_2\biggr\},\\ g(\omega) &= C_1\cosh\biggl\{-\lambda_1\int^{\omega}|\varphi(z)|^{k_1}dz-\tilde{\lambda}_2C_1^{2k_2}\omega+C_2\biggr\},\end{aligned}$$

where $C_2=\text{const}$.

Substitution of the above formulae into the first equation of system (2.7.8) gives rise to the ODE for $\varphi(\omega)$

$$\ddot{\varphi}=\{\mu_1|\varphi|^{k_1}+\tilde{\mu}_2C_1^{2k_2}\}^2\varphi.$$

Representing $\varphi(\omega)$ in the form (2.7.10) we come to the following system of ODEs for $\rho(\omega),\ \theta(\omega)$:

$$\ddot{\rho}-\rho\dot{\theta}^2=\{\mu_1\rho^{k_1}+\tilde{\mu}_2C_1^{2k_2}\}^2\rho,\quad \ddot{\theta}\rho+2\dot{\theta}\dot{\rho}=0.$$

The general solution of the above system is given implicitly

$$\theta=C_3\int^{\omega}\rho^{-1/2}(z)dz,\qquad \int^{\rho(\omega)}\left(2\int a_-(z)dz+C_4\right)^{-1/2}dz=\omega, \tag{2.7.18}$$

where $a_-(z)=\mu_1^2z^{2k_1+1}+2\mu_1\tilde{\mu}_2C_1^{2k_2}z^{k_1+1}+\tilde{\mu}_2^2C_1^{4k_1}z+C_3^2$, C_3, C_4 are constants.

Consequently, the general solution of system of ODEs (2.7.8) under $N=0$, $\lambda=1$ has the form

$$f(\omega) = C_1\sinh\biggl\{-\lambda_1\int^{\omega}\rho^{k_1}(z)dz-\tilde{\lambda}_2C_1^{2k_2}\omega+C_2\biggr\},$$

$$
\begin{aligned}
g(\omega) &= C_1 \cosh\left\{-\lambda_1 \int^{\omega} \rho^{k_1}(z)dz - \tilde{\lambda}_2 C_1^{2k_2}\omega + C_2\right\},\\
\varphi(\omega) &= \rho(\omega)\exp\left\{iC_3 \int^{\omega} \rho^{-/2}(z)dz\right\},
\end{aligned}
$$

function $\rho(\omega)$ being determined by (2.7.18).

4) $N = 1, 2, 3,\ \lambda = -1$.

Solutions of equations (2.7.8) are looked for in the form (2.7.13), whence we get the following system of nonlinear algebraic equations for $C,\ D,\ E$:

$$
\begin{aligned}
&k_1^{-2}(k_1N - k_1 - 1) = -\{\mu_1|E|^{k_1} + \tilde{\mu}_2(D^2 - C^2)^{k_2}\}^2,\\
&(2k_2)^{-1}(1 - 2Nk_2) = \{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(D^2 - C^2)^{k_2}\}DC^{-1},\\
&(2k_2)^{-1} = \{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(D^2 - C^2)^{k_2}\}CD^{-1}.
\end{aligned} \tag{2.7.19}
$$

Analysis of the above equations yields the restriction on the choice of $C,\ D$: $D^2C^{-2} = 1 - 2Nk_2$. Due to this fact equations (2.7.19) are rewritten in the form

$$
\begin{aligned}
&D = \varepsilon C(1 - 2Nk_2)^{1/2},\\
&(1 + k_1 - Nk_1)k_1^{-2} = \{\mu_1|E|k_1 + \tilde{\mu}_2(-2Nk_2C^2)^{k_2}\}^2,\\
&\varepsilon(1 - 2Nk_2)^{1/2}(2k_2)^{-1} = \{\lambda_1|E|^{k_1} + \tilde{\lambda}_2(-2Nk_2C^2)^{k_2}\},
\end{aligned} \tag{2.7.20}
$$

where $\varepsilon = \pm 1,\ k_1 < (N-1)^{-1},\ k_2 < (2N)^{-1}$.

Substitution of the results obtained into Ansatz (2.7.4) gives the following classes of exact solutions of the nonlinear Dirac-d'Alembert equations (1.4.8), (2.7.7) :

<u>the case of arbitrary $k_1 \in \mathbb{R}^1,\ k_2 \in \mathbb{R}^1$</u>

$$
\begin{aligned}
\psi_1(x) &= \left\{i\cos\left(\lambda_1 \int^{x_0} \rho^{k_1}(z)dz + \lambda_2(\bar{\chi}\chi)^{k_2}x_0 + C_2\right)\right.\\
&\quad \left. +\gamma_0 \sin\left(\lambda_1 \int^{x_0} \rho^{k_1}(z)dz + \lambda_2(\bar{\chi}\chi)^{k_2}x_0 + C_2\right)\right\}\chi,\\
u_1(x) &= \rho(x_0)\exp\left\{iC_3 \int^{x_0} \Big(\rho(z)\Big)^{-1/2}dz\right\},
\end{aligned}
$$

where $C_2,\ C_3$ are arbitrary real constants, function $\rho=\rho(\omega)$ is determined by formula (2.7.12) under $C_1=1$:

$$\begin{aligned}\psi_2(x) &= \Bigg\{i\cosh\Bigg(-\lambda_1\int^{\omega}\rho^{k_1}(z)dz-\lambda_2(\bar\chi\chi)^{k}_{2}\omega+C_2\Bigg)\\ &\quad+(\gamma_\mu\partial_\mu\omega)\sinh\Bigg(-\lambda_1\int^{\omega}\rho^{k_1}(z)dz-\lambda_2(\bar\chi\chi)^{k_2}\omega+C_2\Bigg)\Bigg\}\chi,\\ u_2(x) &= \rho(\omega)\exp\Bigg\{iC_3\int^{\omega}\Big(\rho(z)\Big)^{-1/2}dz\Bigg\},\end{aligned}$$

where $C_2,\ C_3$ are arbitrary real constants, function $\rho=\rho(\omega)$ is determined by formula (2.7.18) under $C_1=1$, $\omega=\omega(x)$ is given by one of the formulae listed in (2.1.88);

<u>the case $k_1>1/2,\ k_2>1/6$</u>

$$\begin{aligned}\psi_3(x) &= \omega^{-1/4k_2}\{\varepsilon i(6k_2-1)^{1/2}+\gamma_\mu\partial_\mu\omega\}\chi,\\ u_3(x) &= E\omega^{-1/2k_1},\end{aligned}$$

where $E,\ \chi$ are defined by (2.7.16) under $C=1,\ N=3,\ \omega=\omega(x)$ is given by (2.1.87);

<u>the case $k_1>1,\ k_2>1/4$</u>

$$\begin{aligned}\psi_4(x) &= \omega^{-1/4k_2}\{\varepsilon i(4k_2-1)^{1/2}+\gamma_\mu\partial_\mu\omega\}\chi,\\ u_4(x) &= \omega^{-1/2k_1},\end{aligned}$$

where $E,\ \chi$ are defined by (2.7.16) under $C=1,\ N=2,\ \omega=\omega(x)$ is given by (2.1.86);

<u>the case $k_1<1/2,\ \ k_2<1/6$</u>

$$\begin{aligned}\psi_5(x) &= \omega^{-1/4k_2}\{\varepsilon i(1-6k_2)^{1/2}+\gamma_\mu\partial_\mu\omega\}\chi,\\ u_5(x) &= E\omega^{-1/2k_1},\end{aligned}$$

where $E,\ \chi$ are defined by (2.7.20) under $C=1,\ N=3,\ \omega=\omega(x)$ is given by (2.1.91);

<u>the case $k_1<1,\ k_2<1/4$</u>

$$\begin{aligned}\psi_6(x) &= \omega^{-1/4k_2}\{\varepsilon i(1-4k_2)^{1/2}+\gamma_\mu\partial_\mu\omega\}\chi,\\ u_6(x) &= E\omega^{-1/2k_1},\end{aligned}$$

where $E,\ \chi$ are defined by (2.7.20) under $C=1,\ N=2,\ \omega=\omega(x)$ is given by (2.1.90);

<u>the case $k_1 \in \mathbb{R}^1,\ \ k_2 < 1/2$</u>

$$\begin{aligned}\psi_7(x) &= \omega^{-1/4k_2}\{\varepsilon i(1-2k_2)^{1/2}+\gamma_\mu\partial_\mu\omega\}\chi,\\ u_7(x) &= E\omega^{-1/2k_1},\end{aligned}$$

where $E,\ \chi$ are defined by (2.7.20) under $C=1,\ N=1,\ \omega=\omega(x)$ is given by (2.1.89);

<u>the case $k_1=2,\ k_2=1/2$</u>

$$\begin{aligned}\psi_8(x) &= (1+\theta^2\omega^2)^{-3/2}\{i+\theta\omega\gamma_\mu\partial_\mu\omega\}\chi,\\ u_8(x) &= E(1+\theta^2\omega^2)^{-1/2},\end{aligned}$$

where $\theta,\ E,\ \chi$ are defined by (2.7.17) under $N=2,\ \omega=\omega(x)$ is given by (2.1.86);

<u>the case $k_1=1,\ k_2=1/3$</u>

$$\begin{aligned}\psi_9(x) &= (1+\theta^2\omega^2)^{-2}\{i+\theta\omega\gamma_\mu\partial_\mu\omega\}\chi,\\ u_9(x) &= E(1+\theta^2\omega^2)^{-1},\end{aligned}$$

where $\theta,\ E,\ \chi$ are defined by (2.7.17) under $N=3,\ \omega=\omega(x)$ is given by (2.1.87).

According to Theorem 1.4.1 system of PDEs (1.4.8), (2.7.7) under $k_1=1,\ k_2=1/3$ admits the conformal group $C(1,3)$. Therefore we can apply to solutions $\{\psi_1(x),\, u_1(x)\},\ \{\psi_2(x),\, u_2(x)\},\ \{\psi_3(x),\, u_3(x)\},\ \{\psi_7(x),\, u_7(x)\},\ \{\psi_9(x),\, u_9(x)\}$ with $k_1=1$, $k_2=1/3$ the procedure of generating solutions by means of the four-parameter group of special conformal transformations

$$\begin{aligned}&\psi_{II}(x)=\sigma^{-2}(x)(1-\gamma\cdot\theta\gamma\cdot x)\psi_I(x'),\\ &u_{II}(x)=\sigma^{-1}(x)u_I(x'),\\ &x'_\mu=(x_\mu-\theta_\mu x\cdot x)\sigma^{-1}(x),\end{aligned}\tag{2.7.21}$$

where $\sigma(x)=1-2\theta\cdot x+\theta\cdot\theta x\cdot x$, θ_μ are constants.

The above formulae are obtained from (1.4.13) with the help of Theorem 2.4.1.

The nonlinear functions $\lambda_1|u|^{k_1} + \lambda_2(\bar\psi\psi)^{k_2}$, $\mu_1|u|^{k_1} + \mu_2(\bar\psi\psi)^{k_2}$ can be interpreted as "masses" of the spinor ($M(\psi)$) and scalar ($M(u)$) particles created because of interaction of these particles. As straightforward computation shows, the following relations hold

$$\Big(M(\psi)/M(u)\Big)^2 = (1/4)k_1^2(2Nk_2-1)k_2^{-2}(Nk_1-k_1-1)^{-1},$$

where the cases $N=2,\ N=3$ correspond to the solutions $\{\psi_4,\, u_4\}$, $\{\psi_3,\, u_3\}$;

$$\Big(M(\psi)/M(u)\Big)^2 = (1/4)k_1^2(2Nk_2-1)k_2^{-2}(Nk_1-k_1-1)^{-1},$$

where the cases $N=1,\ N=2,\ N=3$ correspond to the solutions $\{\psi_7,\, u_7\}$, $\{\psi_6,\, u_6\}$, $\{\psi_5,\, u_5\}$;

$$\Big(M(\psi)/M(u)\Big)^{-2} = (N+1)(N-1)^{-1},$$

where the cases $N=2,\ N=3$ correspond to the solutions $\{\psi_8,\, u_8\}$, $\{\psi_9,\, u_9\}$.

Consequently, in spite of the fact that "masses" of the spinor and scalar particles described by equations (1.4.8), (2.7.7) are variable their ratio is the constant determined by the exponents $k_1,\ k_2$ and by some discrete parameter N. Thus, the above relations can be interpreted as the formulae for the mass spectrum of the spinor and scalar fields. It is worth noting that the discrete parameter N arises because of the fact that the nonlinear differential operator $\pm\omega^2\Box$ has the discrete spectrum $N=0,1,2,3$ on the set of solutions of the equation $(\partial_\mu\omega)(\partial^\mu\omega)=\pm1$ (see Section 2.1).

The solutions $\{\psi_3(x),\, u_3(x)\} - \{\psi_9(x),\, u_9(x)\}$ vanish at the infinity under positive $k_1,\ k_2$ and besides they have a non-integrable singularity [151].

Using the fact that system (1.4.8), (2.7.7) under $\lambda_1=\mu_2=0$ splits into the nonlinear Dirac and d'Alembert equations we can get from $\{\psi_1(x),\, u_1(x)\} - \{\psi_9(x),\, u_9(x)\}$ their exact solutions by putting $\lambda_1=0, \mu_2=0$. In particular, the solutions $\{\psi_2(x),\, u_2(x)\}$, $\{\psi_5(x),\, u_5(x)\}$, $\{\psi_6(x),\ u_6(x)\}$ give rise to new solutions of the nonlinear Dirac equation which differ from those constructed in Section 2.4.

In conclusion we will say a few words about exact solutions of the conformally-invariant system of PDEs

$$\begin{aligned} &\partial_\mu\partial^\mu u = \lambda_3 u^3 - \lambda_1\bar\psi\psi,\\ &i\gamma_\mu\partial_\mu\psi = \{\lambda_1 u + \lambda_2(\bar\psi\psi)^{1/3}\}\psi, \end{aligned} \tag{2.7.22}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are constants, obtained in [20, 21] with the help of Heisenberg Ansatz (2.7.1), (2.7.2). Since Ansatz (2.7.1) is a particular case of Ansatz

(2.7.4) (under $\omega(x) = x\cdot x$), the above mentioned solutions can be constructed within the framework of our approach. In particular, functions $\{\psi_3(x),\ u_3(x)\}$, $\{\psi_9(x),\ u_9(x)\}$ with $k_1 = 1,\ k_2 = 1/3$ satisfy system of PDEs (2.7.22) provided the constants $E,\ \chi^\mu,\ \theta$ satisfy the algebraic relations

$$\lambda_1 E + \lambda_2 2^{1/3}(\bar{\chi}\chi)^{1/3} = 3\varepsilon/2, \quad -2\lambda_1(\bar{\chi}\chi) + \lambda_3 E^3 = 3$$

and

$$\lambda_1 E + \lambda_2(\bar{\chi}\chi)^{1/3} = 4\theta, \quad \lambda_1(\bar{\chi}\chi) - \lambda_3 E^3 = 8\theta^2,$$

correspondingly.

2.8. Exact solutions of the nonlinear electrodynamics equations

Let us carry out reduction of Poincaré-invariant equations (1.4.7) for the spinor and vector fields using Ansätze constructed in Section 2.6. Substitution of $P(1,3)$-ungenerable Ansätze for the spinor field (we recall that these are obtained by making the change

$$\begin{aligned} &\gamma_0 \to \gamma\cdot a, \quad \gamma_1 \to \gamma\cdot b, \quad \gamma_2 \to \gamma\cdot c, \quad \gamma_3 \to \gamma\cdot d,\\ &x_0 \to a\cdot z, \quad x_1 \to b\cdot z, \quad x_2 \to c\cdot z, \quad x_3 \to d\cdot z \end{aligned} \tag{2.8.1}$$

in $P(1,3)$-invariant Ansätze (2.2.8)) and $P(1,3)$-ungenerable Ansätze for the vector field (2.6.7), (2.6.8) into system of PDEs (1.4.7), (1.4.18) yields after rather cumbersome computations 27 systems of ODEs for functions $\varphi(\omega)$, $B_\mu(\omega)$. Systems of ODEs for $\varphi(\omega)$ are obtained from (2.3.5) if we replace $\gamma_\mu,\ x_\mu$ by the expressions given in (2.8.1) and put

$$R = \{\gamma\cdot B(f_1 + f_2\gamma_4) + f_3 + f_4\gamma_4\}\varphi,$$

where$f_1, \ldots, f_4$ are arbitrary smooth functions of

$$\bar{\varphi}\varphi, \quad \bar{\varphi}\gamma_4\varphi, \quad \bar{\varphi}\gamma\cdot B\varphi, \quad \bar{\varphi}\gamma_4\gamma\cdot B\varphi, \quad \varphi^T\gamma_0\gamma_2\gamma\cdot B\varphi, \quad B\cdot B. \tag{2.8.2}$$

Reduced systems of ODEs for the vector field are written in the following unified form:

$$\begin{aligned} k_{\mu\gamma}\ddot{B}^\gamma + l_{\mu\gamma}\dot{B}^\gamma + m_{\mu\gamma}B^\gamma \ &= \ g_1 B_\mu + g_2\bar{\varphi}\gamma_\mu\varphi + g_3\bar{\varphi}\gamma_4\gamma_\mu\varphi\\ &\quad + g_4\varphi^T\gamma_0\gamma_2\gamma_\mu\varphi, \quad \mu = 0,\ldots,3, \end{aligned} \tag{2.8.3}$$

where $g_1, \dots, g_4$ are arbitrary smooth functions of the variables (2.8.2) and $k_{\mu\gamma},\ l_{\mu\gamma},\ m_{\mu\gamma}$ are functions of ω listed below

1) $k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0;$

2) $k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0;$

3) $k_{\mu\gamma} = k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0;$

4) $k_{\mu\gamma} = 4g_{\mu\gamma}\omega - a_\mu a_\gamma(\omega+1)^2 - (a_\mu d_\gamma + d_\mu a_\gamma)(\omega^2-1) - d_\mu d_\gamma(\omega-1)^2,$
$l_{\mu\gamma} = 4[g_{\mu\gamma} + (a_\mu d_\gamma - a_\gamma d_\mu)] - 2[a_\mu(\omega+1) + d_\mu(\omega-1)]k_\gamma,$
$m_{\mu\gamma} = 0;$

5) $k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -b_\mu k_\gamma, \quad m_{\mu\gamma} = 0;$

6) $k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma)/\alpha^2;$

7) $k_{\mu\gamma} = -g_{\mu\gamma} - (\alpha k_\gamma e^{-\omega/\alpha} - c_\gamma)(\alpha k_\mu e^{-\omega/\alpha} - c_\mu), \quad l_{\mu\gamma} = (2/\alpha)(a_\mu d_\gamma$
$-a_\gamma d_\mu) + (\alpha k_\mu e^{-\omega/\alpha} - c_\mu)k_\gamma e^{-\omega/\alpha}, \quad m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma)/\alpha^2;$

8) $k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma), \quad m_{\mu\gamma} = -(b_\mu b_\gamma)/\omega;$

9) $k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = (b_\mu b_\gamma + c_\mu c_\gamma)/\alpha^2;$

10) $k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma)/\alpha^2;$

11) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 2(c_\mu b_\gamma - b_\mu c_\gamma), \quad m_{\mu\gamma} = 0;$

12) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -(k_\mu k_\gamma)/\omega, \quad m_{\mu\gamma} = 0;$

13) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -(k_\mu k_\gamma)/\omega, \quad m_{\mu\gamma} = -(k_\mu k_\gamma)/(\alpha^2\omega^2);$

14) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma;$

15) $k_{\mu\gamma} = -4g_{\mu\gamma} - 4b_\mu b_\gamma, \quad m_{\mu\gamma} = l_{\mu\gamma} = 0;$

16) $k_{\mu\gamma} = -4(1+\alpha^2)g_{\mu\gamma} - 4(c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma), \quad m_{\mu\gamma} = l_{\mu\gamma} = 0;$

17) $k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma),$
$m_{\mu\gamma} = -(1/\omega)[(a_\mu a_\gamma - d_\mu d_\gamma)\alpha^{-2} + b_\mu b_\gamma];$

18) $k_{\mu\gamma} = 4[g_{\mu\gamma}\omega - (a_\mu - d_\mu)(a_\gamma - d_\gamma)], \quad l_{\mu\gamma} = 4[g_{\mu\gamma} + (a_\mu d_\gamma - a_\gamma d_\mu)$
$+\alpha(b_\mu c_\gamma - c_\mu b_\gamma)] - 2(a_\mu - d_\mu)k_\gamma, \quad m_{\mu\gamma} = 0;$

19) $k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4g_{\mu\gamma} - 4c_\mu c_\gamma - 2c_\mu k_\gamma \omega^{1/2};$
$m_{\mu\gamma} = -(b_\mu b_\gamma)\omega^{-1};$

20) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -(2k_\mu k_\gamma)/\omega, \quad m_{\mu\gamma} = 0;$

21) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -k_\mu k_\gamma(2\omega+\beta)[\omega(\omega+\beta) - \alpha]^{-1},$
$m_{\mu\gamma} = -k_\mu k_\gamma(\alpha-1)^2[\omega(\omega+\beta) - \alpha]^{-2};$

22) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -k_\mu k_\gamma(2\omega+\beta)[\omega(\omega+\beta)]^{-1},$

$m_{\mu\gamma} = -k_\mu k_\gamma[\omega(\omega+\beta)]^{-2};$

23) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -k_\mu k_\gamma(2\omega+1)[\omega(\omega+\beta)]^{-1}, \quad m_{\mu\gamma} = 0;$

24) $k_{\mu\gamma} = 4\omega g_{\mu\gamma} - (k_\mu\omega + a_\mu - d_\mu)(k_\gamma\omega + a_\gamma - d_\gamma), \quad l_{\mu\gamma} = 6g_{\mu\gamma} + 4(a_\mu d_\gamma - a_\gamma d_\mu) - 3(k_\mu\omega + a_\mu - d_\mu)k_\gamma, \quad m_{\mu\gamma} = -k_\mu k_\gamma;$

25) $k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \beta k_\mu)(c_\gamma - \beta k_\gamma), \quad l_{\mu\gamma} = 2(\beta k_\mu - c_\mu)k_\gamma, \quad m_{\mu\gamma} = -k_\mu k_\gamma;$

26) $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = (c_\mu b_\gamma - b_\mu c_\gamma + 2k_\mu k_\gamma)/\omega, \quad m_{\mu\gamma} = (c_\mu b_\gamma - b_\mu c_\gamma)/\omega;$

27) $k_{\mu\gamma} = 4\omega g_{\mu\gamma} - (a_\mu - d_\mu + k_\mu\omega)(a_\gamma - d_\gamma + k_\gamma\omega), \quad l_{\mu\gamma} = 4[2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - k_\mu k_\gamma\omega - (a_\mu a_\gamma - d_\mu d_\gamma)], \quad m_{\mu\gamma} = -2k_\mu k_\gamma.$

Integration of the above systems of ODEs even under specific F, R_μ is an extremely hard problem. So it is not surprising that up to now there is practically no papers devoted to construction of exact solutions of the Maxwell-Dirac equations (1.4.1).

The fact that ODEs obtained are integrable with some specific F, R_μ is a consequence of their nontrivial symmetry. Using Theorem 2.3.1 we can prove that these systems admit invariance algebras which are isomorphic to algebras (2.3.13).

In the present section we will construct multi-parameter families of exact solutions of classical electrodynamics equations (1.4.1) and of the system of nonlinear PDEs

$$\begin{aligned} &(i\gamma_\mu\partial_\mu - e\gamma_\mu A^\mu)\psi(x) = 0, \\ &\partial_\nu\partial^\nu A_\mu - \partial^\mu\partial_\nu A_\nu = -e\bar{\psi}\gamma_\mu\psi + \lambda A_\mu A_\nu A^\nu, \end{aligned} \tag{2.8.4}$$

where e, λ are constants.

1. Exact solutions of the classical electrodynamics equations. We have made an observation that integrable cases of the systems of ODEs obtained by means of reduction of (2.4.1) with the help of $P(1,3)$-ungenerable Ansätze for the spinor and vector fields give rise to the exact solutions of system of nonlinear PDEs (1.4.1) of the form

$$\begin{aligned} &\psi(x) = (\gamma\cdot a + \gamma\cdot d)\varphi(\omega_1, \omega_2, \omega_3), \\ &A_\mu(x) = (a_\mu + d_\mu)u(\omega_1, \omega_2, \omega_3), \end{aligned} \tag{2.8.5}$$

where $\varphi(\vec{\omega})$ is a four-component complex-valued function, $u(\vec{\omega})$ is a scalar real-valued function; $\omega_1 = b\cdot x$, $\omega_2 = c\cdot x$, $\omega_3 = a\cdot x + d\cdot x$.

Formulae (2.8.5) imply the following method of constructing particular solutions of equation (1.4.1): not to fix *a priori* the functions φ, u in (2.8.5) but to consider them as the new dependent variables. Such an approach proved to be very efficient because it enabled us to obtain exact solutions of the classical electrodynamics equations containing arbitrary functions [155]. Substituting Ansatz (2.8.5) into (1.4.1) and taking into account the identities

$$(\gamma\cdot a+\gamma\cdot d)^2 = a\cdot a + d\cdot d = 0,$$
$$\bar\varphi(\gamma\cdot a+\gamma\cdot d)\gamma_\mu(\gamma\cdot a+\gamma\cdot d)\varphi = 2(a_\mu+d_\mu)\bar\varphi(\gamma\cdot a+\gamma\cdot d)\varphi$$

we come to the system of two-dimensional PDEs for $\varphi(\vec\omega)$, $u(\vec\omega)$

$$\gamma\cdot b\,\varphi_{\omega_1} + \gamma\cdot c\,\varphi_{\omega_2} - im\varphi = 0, \tag{2.8.6}$$
$$u_{\omega_1\omega_1} + u_{\omega_2\omega_2} = 2e\bar\varphi(\gamma\cdot a+\gamma\cdot d)\varphi, \tag{2.8.7}$$

where $\varphi_{\omega_i} = \partial\varphi/\partial\omega_i$, $u_{\omega_i\omega_i} = \partial^2 u/\partial\omega_i^2$, $i=1,2$.

Let us emphasize that in the above equations there is no differentiation with respect to the variable ω_3. Consequently, functions φ, u contain ω_3 as a parameter.

The general solution of PDE (2.8.7) is given by the d'Alembert formula for the two-dimensional Poisson equation [61]

$$\begin{aligned} u(\vec\omega) &= w(z,\omega_3) + w(z^*,\omega_3) \\ &\quad -ie\int^{\omega_2}\int_{\omega_1 - i(\omega_2-\tau)}^{\omega_1 + i(\omega_2-\tau)} \bar\varphi(\xi,\eta)(\gamma\cdot a+\gamma\cdot d)\varphi(\xi,\eta)d\xi d\eta, \end{aligned} \tag{2.8.8}$$

where w is an arbitrary analytical with respect to the variable $z=\omega_1+i\omega_2$ function.

Consequently, the problem of construction of exact solutions of system of nonlinear PDEs (1.4.1) is reduced via Ansatz (2.8.5) to integration of the two-dimensional linear Dirac equation (2.8.6). Using the Fourier transform we can obtain its general solution in the form of the Fourier integral [35, 61] but we restrict ourselves to the cases when it is possible to construct exact solutions in explicit form.

Choosing the eigenfunction of the Hermitian operator $-i\partial_{\omega_1}$ as a particular solution of equation (2.8.6) yields

$$\varphi(\vec\omega) = \exp\{i\lambda\omega_1 + i\gamma\cdot c(\lambda\gamma\cdot b - m)\omega_2\}\varphi_0\,(\omega_3), \tag{2.8.9}$$

where $\varphi_0 \in C^1(\mathbb{R}^1, \mathbb{C}^4)$. Imposing on solution (2.8.9) the condition of 2π-periodicity with respect to ω_1 we get

$$\lambda = \lambda_n = 2\pi n, \quad n \in \mathbb{Z}. \tag{2.8.10}$$

Substituting formulae (2.8.9), (2.8.10) into (2.8.8) and computing the integral we arrive at the explicit expression for $u(\vec{\omega})$

$$\begin{aligned} u(\vec{\omega}) \;=\; & (1/2)(m^2 + \lambda_n^2)^{-1}\{\tau_1 \cosh[2(m^2 + \lambda_n^2)^{1/2}\omega_2] \\ & + \tau_2 \sinh[2(m^2 + \lambda_n^2)^{1/2}\omega_2]\} + w(z,\, \omega_3) + w(z^*,\, \omega_3). \end{aligned} \tag{2.8.11}$$

Here $z = \omega_1 + i\omega_2$, $\tau_1 = e\bar{\varphi}_0(\gamma \cdot a + \gamma \cdot d)\varphi_0$, $\tau_2 = ie(m^2 + \lambda_n^2)^{-1/2}\ \bar{\varphi}_0 \times(\gamma \cdot a + \gamma \cdot d)\ (\lambda_n \gamma \cdot b - m)\varphi_0$.

Substitution of formulae (2.8.9), (2.8.11) into Ansatz (2.8.5) gives a multi-parameter family of the exact solutions of the classical electrodynamics equations (1.4.1) containing three arbitrary functions:

$$\begin{aligned} \psi(x) \;=\; & (\gamma \cdot k)\exp\{i\lambda_n b \cdot x + i\gamma \cdot c(\lambda_n \gamma \cdot b - m)c \cdot x\}\varphi_0(k \cdot x) \\ & \equiv \psi^{(n)}(x), \\ A_\mu(x) \;=\; & k_\mu \Big\{ w(z,\, k \cdot x) + w(z^*,\, k \cdot x) \\ & + (1/2)(m^2 + \lambda_n^2)^{-1}\{\tau_1 \cosh[2(m^2 + \lambda_n^2)^{1/2} c \cdot x] \\ & + \tau_2 \sinh[2(m^2 + \lambda_n^2)^{1/2} c \cdot x]\}\Big\} \equiv A_\mu^{(n)}(x), \end{aligned} \tag{2.8.12}$$

where $k_\mu = a_\mu + d_\mu$.

Similarly, if we choose a particular solution of equation (2.8.6) in the form

$$\begin{aligned} \varphi(\vec{\omega}) \;=\; & (\omega_1^2 + \omega_2^2)^{-1/4}\exp\{-(1/2)(\gamma \cdot b)(\gamma \cdot c)\arctan(\omega_1/\omega_2)\} \\ & \times \exp\{-im(\gamma \cdot c)(\omega_1^2 + \omega_2^2)^{1/2}\}\varphi_0\,(\omega_3), \end{aligned}$$

where $\varphi_0 \in C^1(\mathbb{R}^1, \mathbb{C}^4)$, then formulae (2.8.5), (2.8.8) give rise to the following family of exact solutions:

$$\begin{aligned} \psi(x) \;=\; & |z|^{-1/2}(\gamma \cdot k)\exp\{-(1/2)(\gamma \cdot b)(\gamma \cdot c)\arctan[(b \cdot x) \\ & \times (c \cdot x)^{-1}]\}\exp\{-im\gamma \cdot c\,|z|\}\varphi_0\,(k \cdot x), \\ A_\mu(x) \;=\; & k_\mu \Bigg\{ w(z,\, k \cdot x) + w(z^*,\, k \cdot x) \\ & + \int^{|z|} [\tau_1 \sinh(2my) + \tau_2 \cosh(2my)]y^{-1}dy \Bigg\}. \end{aligned} \tag{2.8.13}$$

In the above formulae w is an arbitrary analytic with respect to $z = b\cdot x + ic\cdot x$ function, $|z|^2 = zz^*$ and

$$\tau_1 = -2e\bar{\varphi}_0(\gamma\cdot k)\varphi_0, \quad \tau_2 = 2ie\bar{\varphi}_0(\gamma\cdot k)(\gamma\cdot c)\varphi_0.$$

We will consider the solution (2.8.12) in more detail putting $w = 0$, $\varphi_0 = \exp\{-\alpha^2(k\cdot x)^2\}\chi$, where χ is an arbitrary constant four-component column, $\alpha = \text{const}$. A direct check shows that the identities

$$\begin{aligned} &-\partial_\mu\partial^\mu A_\nu^{(n)} = 4(m^2 + 4\pi^2 n^2)A_\nu^{(n)}, \quad \partial_\mu A_\mu^{(n)} = 0, \\ &-\partial_\mu\partial^\mu\psi^{(n)} = m^2\psi^{(n)}, \end{aligned} \tag{2.8.14}$$

where $n \in \mathbb{Z}$, $\partial_\mu = \partial/\partial x_\mu$, $\mu = 0,\ldots,3$, hold. The operator $-\partial_\mu\partial^\mu$ is one of the Casimir operators of the Poincaré algebra (see the Appendix 1). Its eigenvalues are interpreted as masses of particles described by the corresponding motion equations. If such an interpretation is extended to a nonlinear case, then relations (2.8.14) can be treated as follows: interaction of the spinor and vector fields according to nonlinear equations (1.4.1) gives rise to the massive vector field $A_\mu^{(n)}(x)$ with the mass $M_n = 2(m^2 + 4\pi^2 n^2)^{1/2}$, $n \in \mathbb{Z}$ (in other words, the nonlinear interaction of the fields $\psi(x)$, $A_\mu(x)$ generates the mass spectrum). Let us emphasize that the effect described is nonlinear because in the case of the linear Maxwell equations the Casimir operator $\partial_\mu\partial^\mu$ has the zero eigenvalue (this is seen from (2.8.12), where $A_\mu^{(n)} = 0$ under $e = 0$).

Since solutions (2.8.12), (2.8.13) depend analytically on m, solutions of the massless classical electrodynamics equations are obtained from (2.8.12), (2.8.13) by putting $m = 0$.

This case deserves a special consideration because the invariance group of equations (1.4.1) under $m = 0$ is the 15-parameter conformal group (Theorem 1.4.2). The general solution of the two-dimensional Dirac equation under $m = 0$ has been constructed in [155]

$$\varphi(\vec{\omega}) = (\gamma\cdot b + i\gamma\cdot c)\varphi_1(z,\omega_3) + (\gamma\cdot b - i\gamma\cdot c)\varphi_2(z^*,\omega_3), \tag{2.8.15}$$

where φ_1, φ_2 are arbitrary four-component functions whose components are analytical functions of $z = b\cdot x + ic\cdot x$ and $z^* = b\cdot x - ic\cdot x$, respectively.

Substitution of (2.8.15) into (2.8.8) yields the following expression for $u(\vec{\omega})$:

$$\begin{aligned} u(\vec{\omega}) \quad = \quad & w(z,\omega_3) + w(z^*,\omega_3) \\ & + e\left\{ z^* \int g_1(z,\omega_3)dz + z\int g_2(z^*,\omega_3)dz^* \right\}, \end{aligned}$$

where $g_1 = \bar{\varphi}_1(\gamma \cdot k)(1 - i\gamma \cdot b\gamma \cdot c)\varphi_2,\ g_2 = \bar{\varphi}_2(\gamma \cdot k)(1 + i\gamma \cdot b\gamma \cdot c)\varphi_1$.

Substituting the above formulae into the Ansatz (2.8.5) we come to the multi-parameter family of the exact solutions which contains five arbitrary complex-valued functions

$$\begin{aligned}\psi(x) &= (\gamma \cdot k)\{(\gamma \cdot b + i\gamma \cdot c)\varphi_1(z,\, k \cdot x) + (\gamma \cdot b - i\gamma \cdot c) \\ &\quad \times\varphi_2(z^*,\, k \cdot x)\}, \\ A_\mu(x) &= k_\mu\Bigg\{w(z,\, k \cdot x) + w(z^*,\, k \cdot x) \\ &\quad +e\left(z^* \int g_1(z,\, k \cdot x)dz + z \int g_2(z^*,\, k \cdot x)dz^*\right)\Bigg\}.\end{aligned} \tag{2.8.16}$$

To obtain $C(1,3)$-ungenerable family of solutions of system of PDEs (1.4.1) with $m = 0$ we employ the solution generation procedure. The formulae for generating solutions of the classical electrodynamics equations (1.4.1) by the four-parameter special conformal transformation group have been obtained in [133]

$$\begin{aligned}\psi_{II}(x) &= \sigma^{-2}(x)(1 - \gamma \cdot x\gamma \cdot \theta)\psi_I(x'), \\ A_{II\mu}(x) &= \sigma^{-2}(x)\{g_{\mu\nu}\sigma(x) + 2(x_\nu\theta_\mu - x_\mu\theta_\nu) &(2.8.17)\\ &\quad +2\theta \cdot x x_\mu\theta_\nu - x \cdot x\theta_\mu\theta_\nu - \theta \cdot \theta x_\mu x_\nu)\}A_I^\nu(x'), &(2.8.18)\end{aligned}$$

where $x'_\mu = (x_\mu - \theta_\mu x \cdot x)\sigma^{-1}(x)$, $\sigma(x) = 1 - 2\theta \cdot x + \theta \cdot \theta\ x \cdot x$, θ_μ are arbitrary real constants.

Substitution of expressions (2.8.16) into (2.8.17) gives rise to the $C(1,3)$ - ungenerable family of exact solutions of system (1.4.1) with $m = 0$. We omit the corresponding formulae because of their awkwardness.

2. Exact solutions of system of nonlinear PDEs (2.8.4). To obtain exact solutions of equations (2.8.4) we apply the Ansatz

$$\begin{aligned}\psi(x) &= (\gamma \cdot a - \gamma \cdot d)\exp\{if(k \cdot x)\}\chi, \\ A_\mu(x) &= (a_\mu - d_\mu)g_1(k \cdot x) + k_\mu g_2(k \cdot x).\end{aligned} \tag{2.8.19}$$

Here $\{f, g_1, g_2\} \subset C^1(\mathbb{R}^1, \mathbb{R}^1),\ \chi$ is an arbitrary four-component constant column.

The Ansatz (2.8.19) reduces equations (2.8.4) to the system of three ODEs for $f(\omega),\ g_1(\omega),\ g_2(\omega)$

$$\dot{f} = -eg_2, \quad \ddot{g}_1 = -2\lambda g_1 g_2^2, \quad g_1^2 g_2 = (e/2\lambda)\bar{\chi}(\gamma \cdot a - \gamma \cdot d)\chi. \tag{2.8.20}$$

On eliminating the function g_2 from the second equation we get the second-order ODE for g_1

$$\ddot{g}_1 = -(\tau^2/\lambda)g_1^{-3}, \tag{2.8.21}$$

where $\tau = 2^{-1/2}e\bar{\chi}(\gamma\cdot a - \gamma\cdot d)\chi$.

The above equation is integrated in elementary functions, its general solution having the form [197]

$$g_1(\omega) = \varepsilon C_1^{-1/2}\Big((C_1\omega + C_2) - \tau^2/\lambda\Big)^{1/2}. \tag{2.8.22}$$

In addition, ODE (2.8.21) with $\lambda > 0$ possesses the one-parameter family of singular solutions

$$g_1(\omega) = \varepsilon(2|\tau||\lambda|^{-1/2}\omega + C_2)^{1/2}. \tag{2.8.23}$$

In (2.8.22), (2.8.23) $C_1,\ C_2$ are real constants, $\varepsilon = \pm 1$.

Substituting formulae (2.8.22), (2.8.23) into the second equation of system (2.8.20) yields

$$g_2(\omega) = C_1\tau\lambda^{-1}\Big((C_1\omega + C_2)^2 - \tau^2/\lambda\Big)^{-1},$$
$$g_2(\omega) = -\tau|\lambda|^{-1}(2|\tau||\lambda|^{-1/2}\omega + C_2)^{-1}.$$

Integrating the first equation of system (2.8.20) we get the explicit form of the function $f(\omega)$

$$f(\omega) = e(-\lambda)^{1/2}\arctan\Big(\tau^{-1}(-\lambda)^{1/2}(C_1\omega + C_2)\Big),$$
$$f(\omega) = -e|\lambda|^{-1/2}\ln(2\tau|\lambda|^{-1/2}\omega + C_2).$$

Substitution of the above formulae into the Ansatz (2.8.19) gives rise to the multi-parameter families of the exact solutions of system (2.8.4)

<u>the case $\lambda \in \mathbb{R}^1,\ \lambda \neq 0$</u>

$$\begin{aligned}
\psi(x) &= (\gamma\cdot a - \gamma\cdot d)\exp\Big\{-ie(-\lambda)^{1/2}\arctan\Big(\tau^{-1}(-\lambda)^{-1/2} \\
&\quad \times[C_1(k\cdot x) + C_2]\Big)\Big\}\chi, \\
A_\mu(x) &= \varepsilon(a_\mu - d_\mu)C_1^{-1/2}\Big\{(C_1 k\cdot x + C_2)^2 - \tau^2\lambda^{-1}\Big\}^{-1/2} \\
&\quad + C_1\tau\lambda^{-1}k_\mu\Big\{(C_1 k\cdot x + C_2)^2 - \tau^2\lambda^{-1}\Big\}^{-1};
\end{aligned}$$

the case $\lambda < 0$

$$\begin{aligned}
\psi(x) &= (\gamma \cdot a - \gamma \cdot d) \exp\Big\{-ie|\lambda|^{-1/2} \ln\Big(2\tau|\lambda|^{-1/2} k \cdot x + C_2\Big)\Big\}\chi, \\
A_\mu(x) &= \varepsilon(a_\mu - d_\mu)\Big\{2|\tau||\lambda|^{-1/2} k \cdot x + C_2\Big\}^{1/2} - \tau|\lambda|^{-1} k_\mu \\
&\quad \times \Big\{2|\tau||\lambda|^{-1/2} k \cdot x + C_2\Big\}^{-1},
\end{aligned}$$

where $\tau = 2^{-1/2} e\bar{\chi}(\gamma \cdot a - \gamma \cdot d)\chi,\ C_1,\ C_2$ are real constants.

Let us note that the solutions obtained are singular with respect to the coupling constant λ. That is why they cannot be obtained in the framework of the perturbation theory by expanding with respect to a small parameter λ.

5. Exact solutions of the Maxwell-Born-Infeld equations. By the Maxwell-Born-Infeld equations we mean the Maxwell equations

$$\begin{aligned}
&\partial_t \vec{D} = \mathrm{rot}\, \vec{H}, \quad \mathrm{div}\, \vec{D} = 0, \\
&\partial_t \vec{B} = -\mathrm{rot}\, \vec{E}, \quad \mathrm{div}\, \vec{B} = 0
\end{aligned} \tag{2.8.24}$$

supplemented by the constitutive equations suggested by Born and Infeld (see, e.g. [142])

$$\vec{D} = \tau\vec{E} + \tau_1\vec{B}, \quad \vec{H} = \tau\vec{B} - \tau_1\vec{E}. \tag{2.8.25}$$

Here $\vec{E}$, $\vec{H}$ are field intensities, $\vec{B}$, $\vec{D}$ are inductions,

$$\begin{aligned}
\tau &= \{1 + \vec{B}^2 - \vec{E}^2 - (\vec{B}\vec{E})^2\}^{-1/2}, \\
\tau_1 &= (\vec{B}\vec{E})\tau.
\end{aligned}$$

Till now, up to our knowledge, there are no papers containing exact solutions of system (2.8.24), (2.8.25) in explicit form. We will construct multiparameter families of exact solutions of system of nonlinear PDEs (2.8.24), (2.8.25) using the following simple assertion.

Lemma 2.8.1. *The general solution of system of PDEs (2.8.24) is given by the formulae*

$$\begin{aligned}
&\vec{B} = \mathrm{rot}\, \vec{u}, \quad \vec{D} = \mathrm{rot}\, \vec{w}, \\
&\vec{H} = \partial_t \vec{w}, \quad \vec{E} = -\partial_t \vec{u},
\end{aligned} \tag{2.8.26}$$

where $\vec{u} = (u_1, u_2, u_3),\ \vec{w} = (w_1, w_2, w_3)$ are arbitrary smooth vector-functions.

To prove the lemma we make use of the well-known fact that the general solutions of equations

$$\mathrm{div}\, \vec{r} = 0, \quad \mathrm{rot}\, \vec{\rho} = \vec{0}$$

are given by the formulae

$$\vec{r} = \mathrm{rot}\,\vec{R}, \quad \vec{\rho} = \mathrm{grad} R_0,$$

where $R_a,\ R_0$ are arbitrary twice differentiable functions. $\triangleright$

According to Lemma 2.8.1, the Maxwell-Born-Infeld equations are represented in the form (2.8.26), where $\vec{u},\ \vec{w}$ are smooth vector-functions satisfying the first-order system of nonlinear PDEs

$$\begin{aligned} \mathrm{rot}\,\vec{w} &= -\tau\{\partial_t\vec{u} + [(\partial_t\vec{u})(\mathrm{rot}\,\vec{u})]\mathrm{rot}\,\vec{u}\}, \\ \partial_t\vec{w} &= \tau\{\mathrm{rot}\,\vec{u} - [(\partial_t\vec{u})(\mathrm{rot}\,\vec{u})]\partial_t\vec{u}\}, \end{aligned} \tag{2.8.27}$$

with $\tau = \{1 + (\mathrm{rot}\,\vec{u})^2 - (\partial_t\vec{u})^2 - [(\partial_t\vec{u})(\mathrm{rot}\,\vec{u})]^2\}^{-1/2}$.

Thus, the over-determined system of fourteen equations (2.8.24), (2.8.25) for twelve functions $E_a,\ H_a,\ D_a,\ B_a$ is reduced to the system of six nonlinear PDEs for six functions $u_a,\ w_a$.

To construct exact solutions of (2.8.27) we apply the Ansatz [170]

$$\vec{u} = \vec{a}\varphi(t,\ \vec{b}\,\vec{x},\ \vec{c}\,\vec{x}) \equiv \vec{a}\varphi(\omega_0, \omega_1, \omega_2). \tag{2.8.28}$$

Here $\varphi \in C^2(\mathbb{R}^3, \mathbb{R}^1)$; $\vec{a},\ \vec{b},\ \vec{c}$ are arbitrary constant vectors satisfying the conditions

$$\begin{gathered} \vec{a}^{\,2} = \vec{b}^{\,2} = \vec{c}^{\,2} = 1, \\ \vec{a}\,\vec{b} = \vec{b}\,\vec{c} = \vec{c}\,\vec{a} = 0. \end{gathered}$$

Since $\mathrm{rot}\,\vec{u} = -\vec{c}\varphi_{\omega_1} + \vec{b}\varphi_{\omega_2}$, the equality $(\partial_t\vec{u})(\mathrm{rot}\,\vec{u}) = 0$ holds. Consequently, system (2.8.27) takes the form

$$\begin{aligned} \mathrm{rot}\,\vec{w} &= -\tau\vec{a}\varphi_{\omega_0}, \\ \partial_t\vec{w} &= \tau(-\vec{c}\varphi_{\omega_1} + \vec{b}\varphi_{\omega_2}), \end{aligned} \tag{2.8.29}$$

where

$$\tau = (\varphi^2_{\omega_1} + \varphi^2_{\omega_2} - \varphi^2_{\omega_0} + 1)^{-1/2}.$$

The compatibility condition $\partial_t(\mathrm{rot}\,\vec{w}) = \mathrm{rot}\,(\partial_t\vec{w})$ when applied to (2.8.29) yields

$$\partial_t(-\tau\vec{a}\varphi_{\omega_0}) = \mathrm{rot}\,[\tau(-\vec{c}\varphi_{\omega_1} + \vec{b}\varphi_{\omega_2})]$$

or

$$\vec{a}(1 - \varphi_{\omega_\mu}\varphi_{\omega^\mu})^{-3/2}\{(1 - \varphi_{\omega_\mu}\varphi_{\omega^\mu})\Box\varphi + \varphi_{\omega_\mu\omega_\nu}\varphi_{\omega^\mu}\varphi_{\omega^\nu}\} = \vec{0}.$$

Here summation over the repeated indices in the pseudo-Euclidean space $R(1,2)$ is used.

Consequently, provided $\varphi(\omega)$ satisfies the nonlinear scalar PDE

$$(1-\varphi_{\omega_\mu}\varphi_{\omega^\mu})\Box\varphi + \varphi_{\omega_\mu\omega_\nu}\varphi_{\omega^\mu}\varphi_{\omega^\nu} = 0 \tag{2.8.30}$$

with $1-\varphi_{\omega_\mu}\varphi_{\omega^\mu} \neq 0$, formulae (2.8.26), (2.8.28), (2.8.29) give a particular solution of system of nonlinear PDEs (2.2.22), (2.8.25).

Wide classes of exact solutions of nonlinear equation (2.8.30) were constructed in [137]. Inserting these into formulae (2.2.24) and (2.2.26) we get the following multi-parameter families of exact solutions of the Maxwell-Born-Infeld equations:

$$\begin{aligned}
\vec{E} &= -\vec{a}(\dot{h}_1\vec{c}\,\vec{x}+\dot{h}_2),\\
\vec{H} &= (1+h_1^2)^{-1/2}[h_1\vec{b}-(\dot{h}_1\vec{c}\,\vec{x}+\dot{h}_2)\vec{c}],\\
\vec{B} &= h_1\vec{b}-\vec{c}(\dot{h}_1\vec{c}\,\vec{x}+\dot{h}_2),\\
\vec{D} &= -\vec{a}(1+h_1^2)^{-1/2}(\dot{h}_1\vec{c}\,\vec{x}+\dot{h}_2),
\end{aligned}$$

$$\begin{aligned}
\vec{E} &= -(C_1t/\omega)\vec{a}(1+C_2\omega^4)^{-1/2},\\
\vec{H} &= (C_1/\omega)[-\vec{b}(\vec{c}\,\vec{x})+\vec{c}(\vec{b}\,\vec{x})](1+C_2\omega^4-C_1^2)^{-1/2};\\
\vec{B} &= (C_1/\omega)[-\vec{b}(\vec{c}\,\vec{x})+\vec{c}(\vec{b}\,\vec{x})](1+C_2\omega^4)^{-1/2},\\
\vec{D} &= -(C_1t/\omega)\vec{a}(1+C_2\omega^4-C_1^2)^{-1/2},
\end{aligned}$$

$$\begin{aligned}
\vec{E} &= \mp(1/4)\vec{a}\{C_1^{-1}(t-\vec{b}\,\vec{x})^{-1}\coth[C_1(t+\vec{b}\,\vec{x})+C_2]\}^{1/2}\\
&\quad\times\{2C_1(t-\vec{b}\,\vec{x})+\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}\\
&\quad\times\cosh^{-2}[C_1(t+\vec{b}\,\vec{x})+C_2],\\
\vec{H} &= \mp 2^{-3/2}\vec{c}\{2C_1(t-\vec{b}\,\vec{x})-\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}\\
&\quad\times C_1^{-1/2}(t-\vec{b}\,\vec{x})^{-1/2}\{\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}^{-1/2},\\
\vec{B} &= \mp(1/4)\vec{c}\{C_1^{-1}(t-\vec{b}\,\vec{x})^{-1}\coth[C_1(t+\vec{b}\,\vec{x})+C_2]\}^{1/2}\\
&\quad\times\{2C_1(t-\vec{b}\,\vec{x})-\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}\\
&\quad\times\cosh^{-2}[C_1(t+\vec{b}\,\vec{x})+C_2],\\
\vec{D} &= \mp 2^{-3/2}\vec{a}\{2C_1(t-\vec{b}\,\vec{x})+\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}\\
&\quad\times C_1^{-1/2}(t-\vec{b}\,\vec{x})^{-1/2}\{\sinh 2[C_1(t+\vec{b}\,\vec{x})+C_2]\}^{-1/2},
\end{aligned}$$

$$\vec{E} = \mp(1/2)\vec{a}\Big\{2C_3^{-1}+C_2C_3\exp\{C_3(t-\vec{b}\,\vec{x})\}\Big\}$$

$$
\begin{aligned}
&\times\Big\{C_2\exp\{C_3(t-\vec{b}\,\vec{x})\}+2C_3^{-1}(t+\vec{b}\,\vec{x})\Big\}^{-1/2},\\
\vec{H} &= \mp(1/2)\vec{c}\Big\{2C_3^{-1}-C_2C_3\exp\{C_3(t-\vec{b}\,\vec{x})\}\Big\}\\
&\times\Big\{C_2\exp\{C_3(t-\vec{b}\,\vec{x})\}+2C_3^{-1}(t+\vec{b}\,\vec{x})\Big\}^{-1/2},\\
\vec{B} &= \mp(1/2)\vec{c}\Big\{2C_3^{-1}-C_2C_3\exp\{C_3(t-\vec{b}\,\vec{x})\}\Big\}\\
&\times\Big\{C_2\exp\{C_3(t-\vec{b}\,\vec{x})\}+2C_3^{-1}(t+\vec{b}\,\vec{x})\Big\}^{-1/2},\\
\vec{D} &= \mp(1/2)\vec{a}\Big\{2C_3^{-1}+C_2C_3\exp\{C_3(t-\vec{b}\,\vec{x})\}\Big\}\\
&\times\Big\{-C_2\exp\{C_3(t-\vec{b}\,\vec{x})\}+2C_3^{-1}(t+\vec{b}\,\vec{x})\Big\}^{-1/2},
\end{aligned}
$$

where $h_i = h_i(t+\vec{b}\,\vec{x}) \in C^2(\mathbb{R}^1,\mathbb{R}^1)$ are arbitrary functions; $C_1,\ C_2,\ C_3$ are arbitrary real constants; $\omega^2 = \omega_0^2-\omega_1^2-\omega_2^2 \equiv t^2-(\vec{b}\,\vec{x})^2-(\vec{c}\,\vec{x})^2$.

Other classes of exact solutions of system of PDEs (2.8.24), (2.8.25) are obtained by putting

$$\mathrm{rot}\vec{u}=\vec{0},\quad \vec{u}_{tt}=\vec{0} \tag{2.8.31}$$

in (2.8.27).

Resulting from (2.8.31) equations (2.8.27) take the form

$$
\begin{aligned}
&\mathrm{rot}\vec{w} = -\{1-(\mathrm{grad}\varphi)^2\}^{-1/2}\mathrm{grad}\varphi,\\
&\vec{u} = \mathrm{grad}(t\varphi+v),
\end{aligned}
\tag{2.8.32}
$$

where $\{\varphi(\vec{x}), v(\vec{x})\} \subset C^2(\mathbb{R}^3,\mathbb{R}^1)$ are arbitrary functions.

Since $\mathrm{div}\,(\mathrm{rot}\vec{w})=0$, from (2.8.32) it follows that

$$\mathrm{div}\,\{[1-(\mathrm{grad}\,\varphi)^2]^{-1/2}\mathrm{grad}\varphi\}=0,$$

whence

$$[1-(\mathrm{grad}\,\varphi)^2]^{-3/2}\{[1-(\mathrm{grad}\,\varphi)^2]\Delta\varphi+\varphi_{x_ax_b}\varphi_{x_a}\varphi_{x_b}\}=0.$$

The above equation with $(\mathrm{grad}\varphi)^2 \neq 1$ is equivalent to the elliptic analogue of PDE (2.8.30)

$$(1-(\mathrm{grad}\varphi)^2)\Delta\varphi+\varphi_{x_ax_b}\varphi_{x_a}\varphi_{x_b}=0. \tag{2.8.33}$$

In [137] the following two classes of exact solutions of nonlinear PDE (2.8.33) were constructed

$$
\begin{aligned}
\varphi(\vec{x}) &= C_1 \ln\Big\{\Big((\vec{a}\,\vec{x}+C_2)^2+(\vec{b}\,\vec{x}+C_3)^2\Big)^{1/2} \\
&\quad +\Big((\vec{a}\,\vec{x}+C_2)^2+(\vec{b}\,\vec{x}+C_3)^2+C_1^2\Big)^{1/2}\Big\}, \\
\varphi(\vec{x}) &= \int\limits_{\omega_0}^{(\vec{x}^2)^{1/2}} (1+C_1^2\tau^4)^{-1/2}d\tau,
\end{aligned}
$$

where $C_a,\ a=1,2,3,\ \omega_0$ are arbitrary real constants.

Inserting the above formulae into (2.8.26), (2.8.32) we get two multi-parameter families of exact solutions of the Maxwell-Born-Infeld equations

$$
\begin{aligned}
\vec{B} &= 0, \quad \vec{H} = \vec{0}, \\
\vec{D} &= C_1\{\vec{a}(\vec{a}\,\vec{x}+C_2)+\vec{b}(\vec{b}\,\vec{x}+C_3)\}[(\vec{a}\,\vec{x}+C_2)^2+(\vec{b}\,\vec{x}+C_3)^2]^{-1}, \\
\vec{E} &= C_1\{\vec{a}(\vec{a}\,\vec{x}+C_2)+\vec{b}(\vec{b}\,\vec{x}+C_3)\}[(\vec{a}\,\vec{x}+C_2)^2 \\
&\quad +(\vec{b}\,\vec{x}+C_3)^2]^{-1/2}[(\vec{a}\,\vec{x}+C_2)^2+(\vec{b}\,\vec{x}+C_3)^2+C_1^2]^{-1/2};
\end{aligned}
$$

$$
\begin{aligned}
\vec{B} &= \vec{0}, \quad \vec{H} = \vec{0}, \\
\vec{D} &= -(1/C_1)\vec{x}(\vec{x}^{\,2})^{-3/2}, \\
\vec{E} &= -\vec{x}(\vec{x}^{\,2})^{-1/2}\{1+C_1^2(\vec{x}^{\,2})^2\}^{-1/2},
\end{aligned}
$$

where $C_1,\ C_2,\ C_3$ are arbitrary real constants, $C_1\neq 0$.

TWO-DIMENSIONAL SPINOR MODELS

In this chapter nonlinear spinor PDEs with two independent variables x_0, x_1 invariant under infinite-parameter groups are considered. Such a broad symmetry makes it possible to obtain changes of variables which linearize equations considered and to construct their general solutions. Partial linearization of the nonlinear Thirring system of PDEs is carried out.

3.1. Two-dimensional spinor equations invariant under infinite-parameter groups

Invariance of PDEs under study with respect to some infinite-parameter Lie groups makes it possible to construct their exact solutions containing arbitrary functions. Of special interest are two-dimensional equations possessing such a property since many of them can be integrated in closed form. Methods used to construct the general solutions of the two-dimensional d'Alembert [123], Liouville [145], Born-Infeld [145], Monge-Ampère [145, 146], gas dynamics [129], massless Thirring [249, 277] equations are, in fact, based on the unique symmetry of the equations enumerated.

It is worth noting that most of the equations which are integrable by the inverse scattering method also possess broad symmetry. They are invariant under infinite-parameter Lie-Bäcklund groups [7, 190, 233].

We will show that the list of integrable two-dimensional PDEs can be supplemented by the following equations:

$$\Big(i\gamma_\mu\partial_\mu - \lambda\gamma_\mu(\bar{\psi}\gamma^\mu\psi)\Big)\psi = 0; \tag{3.1.1}$$

$$\begin{cases} (i\gamma_\mu\partial_\mu - e\gamma_\mu A^\mu)\psi = 0, \\ \partial_\nu\partial^\nu A_\mu - \partial^\mu\partial_\nu A_\nu = -e\bar{\psi}\gamma_\mu\psi; \end{cases} \tag{3.1.2}$$

$$\Big(i(\gamma_0+\gamma_4)\partial_0 + i\gamma_1\partial_1 - \lambda(\psi^\dagger\psi + \bar{\psi}\gamma_4\psi)^{1/2k}\Big)\psi = 0. \tag{3.1.3}$$

In (3.1.1)–(3.1.3) $\psi = \psi(x_0, x_1)$ is a four-component complex-valued function-column; $A_0(x_0, x_1)$, $A_1(x_0, x_1)$ are real-valued functions; $\mu, \nu = 0, 1$; λ, e are constants.

Dirac matrices are chosen in the form

$$\gamma_0 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}.$$

Let us note that PDE (3.1.1) is a two-dimensional analogue of the Dirac-Heisenberg equation [180, 184], system (3.1.2) is a two-dimensional system of massless classical electrodynamics equations, PDE (3.1.3) is a two-dimensional Galilei-invariant equation for a massless particle with the spin $s = 1/2$ (see also the Section 4.1).

Symmetry properties of equations (3.1.1)–(3.1.3) are described by the following assertions.

Theorem 3.1.1[146, 293]. *System of PDEs (3.1.1) is invariant under the infinite-parameter transformation group*

$$G_\infty = (O_\xi \otimes O_\eta) \otimes \widetilde{G},$$

where O_ξ is the group of linear transformations of the space $(\psi^0, \psi^{0*}, \psi^2, \psi^{2*})$ preserving the quadratic form $|\psi^0|^2 + |\psi^2|^2$, its parameters being arbitrary smooth functions of $\xi = x_0 + x_1$, $|\psi^0|$, $|\psi^2|$; O_η is the group of linear transformations in the space $(\psi^1, \psi^{1*}, \psi^3, \psi^{3*})$ preserving the quadratic form $|\psi^1|^2 + |\psi^3|^2$, its parameters being arbitrary smooth functions of $\eta = x_0 - x_1$, $|\psi^1|$, $|\psi^3|$; the group $\widetilde{G}$ is given by the formulae

$$\begin{aligned} x_0' &= \frac{1}{2}\Bigg(\int^{x_0+x_1} f_0^{-2}(z)dz + \int^{x_0-x_1} f_1^{-2}(z)dz\Bigg), \\ x_1' &= \frac{1}{2}\Bigg(\int^{x_0+x_1} f_0^{-2}(z)dz - \int^{x_0-x_1} f_1^{-2}(z)dz\Bigg), \\ \psi'^0 &= f_0(x_0+x_1)\psi^0, \ \psi'^1 = f_1(x_0-x_1)\psi^1, \\ \psi'^2 &= f_0(x_0+x_1)\psi^2, \ \psi'^3 = f_1(x_0-x_1)\psi^3, \end{aligned} \tag{3.1.4}$$

In (3.1.4) $f_0,\ f_1$ *are arbitrary smooth real-valued functions.*

Proof. After writing component-wise we represent (3.1.1) in the form

$$\begin{aligned}
i(\partial_0 - \partial_1)\psi^0 &= 2\lambda(|\psi^1|^2 + |\psi^3|^2)\psi^0,\\
i(\partial_0 + \partial_1)\psi^1 &= 2\lambda(|\psi^0|^2 + |\psi^2|^2)\psi^1,\\
i(\partial_0 - \partial_1)\psi^2 &= 2\lambda(|\psi^1|^2 + |\psi^3|^2)\psi^2,\\
i(\partial_0 + \partial_1)\psi^3 &= 2\lambda(|\psi^0|^2 + |\psi^2|^2)\psi^3.
\end{aligned}$$

In the cone variables $\xi = x_0 + x_1,\ \eta = x_0 - x_1$ the above system reads

$$\begin{aligned}
i\partial_\eta\psi^0 &= \lambda(|\psi^1|^2 + |\psi^3|^2)\psi^0,\\
i\partial_\xi\psi^1 &= \lambda(|\psi^0|^2 + |\psi^2|^2)\psi^1,\\
i\partial_\eta\psi^2 &= \lambda(|\psi^1|^2 + |\psi^3|^2)\psi^2,\\
i\partial_\xi\psi^3 &= \lambda(|\psi^0|^2 + |\psi^2|^2)\psi^3,
\end{aligned} \tag{3.1.5}$$

the group $\widetilde{G}$ being given by the formulae

$$\begin{aligned}
&\xi' = \int^{\xi} f_0^{-2}(z)dz,\ \eta' = \int^{\eta} f_1^{-2}(z)dz,\\
&\psi'^0 = f_0(\xi)\psi^0,\ \psi'^1 = f_1(\eta)\psi^1,\\
&\psi'^2 = f_0(\xi)\psi^2,\ \psi'^3 = f_1(\eta)\psi^3.
\end{aligned} \tag{3.1.6}$$

Applying to both parts of the first equation of system (3.1.5) the operation of complex conjugation we have

$$-i\partial_\eta\psi^{*0} = \lambda(|\psi^1|^2 + |\psi^3|^2)\psi^{*0},$$

whence

$$\psi^0\partial_\eta\psi^{*0} + \psi^{*0}\partial_\eta\psi^0 = 0$$

or

$$\partial_\eta|\psi^0| = 0.$$

Similarly,

$$\partial_\xi|\psi^1| = 0, \quad \partial_\eta|\psi^2| = 0, \quad \partial_\xi|\psi^3| = 0.$$

From the equalities obtained it follows that system of PDEs (3.1.5) is invariant under the group $O_\xi \otimes O_\eta$.

Let us prove that system (3.1.5) admits transformation group (3.1.6). To this end we make in (3.1.5) the change of variables according to formulae (3.1.6) thus obtaining the following equations:

$$\begin{aligned}
i\partial_{\eta'}\psi'^0 - \lambda(|\psi'^1|^2 + |\psi'^3|^2)\psi'^0 &= f_0 f_1^2\Big(i\partial_\eta\psi^0 - \lambda(|\psi^1|^2 + |\psi^3|^2)\psi^0\Big),\\
i\partial_{\xi'}\psi'^1 - \lambda(|\psi'^0|^2 + |\psi'^2|^2)\psi'^1 &= f_1 f_0^2\Big(i\partial_\xi\psi^1 - \lambda(|\psi^0|^2 + |\psi^2|^2)\psi^1\Big),\\
i\partial_{\eta'}\psi'^2 - \lambda(|\psi'^1|^2 + |\psi'^3|^2)\psi'^2 &= f_0 f_1^2\Big(i\partial_\eta\psi^2 - \lambda(|\psi^1|^2 + |\psi^3|^2)\psi^2\Big),\\
i\partial_{\xi'}\psi'^3 - \lambda(|\psi'^0|^2 + |\psi'^2|^2)\psi'^3 &= f_1 f_0^2\Big(i\partial_\xi\psi^3 - \lambda(|\psi^0|^2 + |\psi^2|^2)\psi^2\Big),
\end{aligned}$$

whence the validity of the theorem follows. $\triangleright$

Theorem 3.1.2. *System of PDEs (3.1.2) is invariant under the infinite-parameter transformation group of the form*

$$G_\infty = (O_\xi \otimes O_\eta) \otimes \widetilde{P}(1,1) \otimes U(1),$$

where $\widetilde{P}(1,1)$ is the extended Poincaré group, U(1) is the group of gauge transformations

$$\begin{aligned}
\psi'^\alpha &= \psi^\alpha \exp\{-ief\},\\
A'_\mu &= A_\mu + \partial^\mu f
\end{aligned}$$

with $f = f(x_0, x_1) \in C^3(\mathbb{R}^2, \mathbb{R}^1)$.

Theorem 3.1.3. *System of PDEs (3.1.3) is invariant under the infinite-parameter transformation group having the following generators:*

under $k \neq 1/2$

$$\begin{aligned}
&P_0 = \partial_0, \quad P_1 = \partial_1,\\
&D_1 = x_0\partial_0 + x_1\partial_1 + k,\\
&D_2 = 2x_0\partial_0 + x_1\partial_1 + k + (1/2)(1 - \gamma_0\gamma_4),\\
&G = w_1(x_0)\partial_1 - (1/2)\dot{w}_1(x_0)(\gamma_0 + \gamma_4)\gamma_1,\\
&Q = (\gamma_0 + \gamma_4)\Big(\gamma_2 w_2(x_0) + \gamma_3 w_3(x_0)\Big);
\end{aligned}$$

under $k = 1/2$

$$\begin{aligned}
\widetilde{A} = {}& w_0(x_0)\partial_0 + \dot{w}_0(x_0)x_1\partial_1 + (1/2)\dot{w}_0(x_0)\\
&-(1/2)\ddot{w}_0(x_0)x_1(\gamma_0 + \gamma_4)\gamma_1,
\end{aligned}$$

$$G = w_1(x_0)\partial_1 - (1/2)\dot{w}_1(x_0)(\gamma_0 + \gamma_4)\gamma_1,$$
$$Q = (\gamma_0 + \gamma_4)\Big(\gamma_2 w_2(x_0) + \gamma_3 w_3(x_0)\Big),$$
$$D = 2x_0\partial_0 + x_1\partial_1 + 1 - (1/2)\gamma_0\gamma_4.$$

Here $w_0, \dots, w_3$ are arbitrary smooth real-valued functions.

Theorem 3.1.2 is proved in the same way as Theorem 3.1.1. To prove Theorem 3.1.3 it is necessary to apply the Lie method.

Let us note that symmetry properties of equations (3.1.1)–(3.1.3) are not exhausted by the invariance under the local symmetry groups described above. As shown in [146] system of PDEs (3.1.1) is invariant under the group of nonlocal (integral) transformations

$$\begin{aligned}
\psi'^0 &= \theta_0\psi^0 \exp\Big\{-i\lambda \int^{x_0 - x_1} \Big((|\theta_1|^2 - 1)|\psi^1|^2 + (|\theta_3|^2 - 1)|\psi^3|^2\Big) d\eta\Big\},\\
\psi'^1 &= \theta_1\psi^1 \exp\Big\{-i\lambda \int^{x_0 + x_1} \Big((|\theta_0|^2 - 1)|\psi^0|^2 + (|\theta_2|^2 - 1)|\psi^2|^2\Big) d\xi\Big\},\\
\psi'^2 &= \theta_2\psi^2 \exp\Big\{-i\lambda \int^{x_0 - x_1} \Big((|\theta_1|^2 - 1)|\psi^1|^2 + (|\theta_3|^2 - 1)|\psi^3|^2\Big) d\eta\Big\},\\
\psi'^3 &= \theta_3\psi^3 \exp\Big\{-i\lambda \int^{x_0 + x_1} \Big((|\theta_0|^2 - 1)|\psi^0|^2 + (|\theta_2|^2 - 1)|\psi^2|^2\Big) d\xi\Big\},
\end{aligned}$$

where $\{\theta_0, \dots, \theta_3\} \subset \mathbb{C}^1$.

3.2. Nonlinear two-dimensional Dirac-Heisenberg equations

In this section we will construct the general solution of system (3.1.1) with the help of the nonlocal linearization method [145, 146]. In other words, a nonlocal change of variables reducing (3.1.1) to a system of linear PDEs will be suggested.

The form of the change of variables is implied by the structure of the group of integral transformations given at the end of the previous section. We introduce new dependent variables $\varphi^0(\xi, \eta), \dots, \varphi^3(\xi, \eta)$ in the following way:

$$\psi^0 = \varphi^0 \exp\Big\{-i\lambda \int \Big(|\varphi^1|^2 + |\varphi^3|^2\Big) d\eta\Big\},$$

$$
\begin{aligned}
\psi^1 &= \varphi^1 \exp\Big\{-i\lambda \int \Big(|\varphi^0|^2 + |\varphi^2|^2\Big) d\xi\Big\}, \\
\psi^2 &= \varphi^2 \exp\Big\{-i\lambda \int \Big(|\varphi^1|^2 + |\varphi^3|^2\Big) d\eta\Big\}, \\
\psi^3 &= \varphi^3 \exp\Big\{-i\lambda \int \Big(|\varphi^0|^2 + |\varphi^2|^2\Big) d\xi\Big\}.
\end{aligned}
\tag{3.2.1}
$$

Substituting (3.2.1) into (3.1.5) we get a system of linear equations for $\varphi^0, \dots, \varphi^3$

$$
\begin{aligned}
&\partial_\eta \varphi^0 = 0, \quad \partial_\xi \varphi^1 = 0, \\
&\partial_\eta \varphi^2 = 0, \quad \partial_\xi \varphi^3 = 0.
\end{aligned}
\tag{3.2.2}
$$

Integration of the above equations yields the following expressions for $\varphi^\mu,\ \mu = 0, \dots, 3$:

$$
\begin{aligned}
&\varphi^0 = U^0(\xi), \quad \varphi^1 = U^1(\eta), \\
&\varphi^2 = U^2(\xi), \quad \varphi^3 = U^3(\eta),
\end{aligned}
\tag{3.2.3}
$$

where $U^\mu \in C^1(\mathbb{R}^1, \mathbb{C}^1)$ are arbitrary functions.

Substitution of formulae (3.2.3) into (3.2.1) with subsequent change of independent variables $\xi \to x_0 + x_1,\ \eta \to x_0 - x_1$ gives the general solution of the nonlinear Dirac-Heisenberg equation (3.1.1)

$$
\begin{aligned}
\psi^0(x) &= U^0(x_0 + x_1) \exp\Big\{-i\lambda \int^{x_0 - x_1} \Big(|U^1|^2 + |U^3|^2\Big) d\tau\Big\}, \\
\psi^1(x) &= U^1(x_0 - x_1) \exp\Big\{-i\lambda \int^{x_0 + x_1} \Big(|U^0|^2 + |U^2|^2\Big) d\tau\Big\}, \\
\psi^2(x) &= U^2(x_0 + x_1) \exp\Big\{-i\lambda \int^{x_0 - x_1} \Big(|U^1|^2 + |U^3|^2\Big) d\tau\Big\}, \\
\psi^3(x) &= U^3(x_0 - x_1) \exp\Big\{-i\lambda \int^{x_0 + x_1} \Big(|U^0|^2 + |U^2|^2\Big) d\tau\Big\}.
\end{aligned}
\tag{3.2.4}
$$

The result obtained enables us to construct in explicit form solution of the classical Cauchy problem for system of PDEs (3.1.1)

$$
\begin{aligned}
&\Big(i\gamma_\mu \partial_\mu - \lambda \gamma_\mu (\bar{\psi}\gamma^\mu \psi)\Big)\psi = 0, \\
&\psi^\alpha(0, x_1) = f^\alpha(x_1),\ x_1 \in \mathbb{R}^1,
\end{aligned}
\tag{3.2.5}
$$

where $f^\alpha \in C^1(\mathbb{R}^1, \mathbb{C}^1)$, $\alpha = 0, \dots, 3$.

Imposing on the solution (3.2.4) the initial conditions of the Cauchy problem (3.2.5) we get functional relations for determination of U^μ, $\mu = 0, \dots, 3$

$$\begin{aligned}
f^0(z) &= U^0(z) \exp\left\{-i\lambda \int^{-z} \left(|U^1|^2 + |U^3|^2\right) d\tau\right\},\\
f^1(z) &= U^1(-z) \exp\left\{-i\lambda \int^{z} \left(|U^0|^2 + |U^2|^2\right) d\tau\right\},\\
f^2(z) &= U^2(z) \exp\left\{-i\lambda \int^{-z} \left(|U^1|^2 + |U^3|^2\right) d\tau\right\},\\
f^3(z) &= U^3(-z) \exp\left\{-i\lambda \int^{z} \left(|U^0|^2 + |U^2|^2\right) d\tau\right\},
\end{aligned}$$

whence it follows

$$\begin{aligned}
U^0(z) &= f^0(z) \exp\left\{i\lambda \int^{-z} \left(|f^1(-\tau)|^2 + |f^3(-\tau)|^2\right) d\tau\right\},\\
U^1(-z) &= f^1(z) \exp\left\{i\lambda \int^{z} \left(|f^0(\tau)|^2 + |f^2(\tau)|^2\right) d\tau\right\},\\
U^2(z) &= f^2(z) \exp\left\{i\lambda \int^{-z} \left(|f^1(-\tau)|^2 + |f^3(-\tau)|^2\right) d\tau\right\},\\
U^3(-z) &= f^3(z) \exp\left\{i\lambda \int^{z} \left(|f^0(\tau)|^2 + |f^2(\tau)|^2\right) d\tau\right\}.
\end{aligned}$$

Substitution of the above equalities into (3.2.4) gives the solution of the Cauchy problem (3.2.5)

$$\begin{aligned}
\psi^0(x) &= f^0(x_1 + x_0) \exp\left\{i\lambda \int_{x_1 + x_0}^{x_1 - x_0} \left(|f^1|^2 + |f^3|^2\right) d\tau\right\},\\
\psi^1(x) &= f^1(x_1 - x_0) \exp\left\{-i\lambda \int_{x_1 - x_0}^{x_1 + x_0} \left(|f^0|^2 + |f^2|^2\right) d\tau\right\},
\end{aligned}$$

$$
\begin{aligned}
\psi^2(x) &= f^2(x_1 + x_0) \exp\left\{ i\lambda \int\limits_{x_1 + x_0}^{x_1 - x_0} \Big(|f^1|^2 + |f^3|^2 \Big) d\tau \right\}, \\
\psi^3(x) &= f^3(x_1 - x_0) \exp\left\{ -i\lambda \int\limits_{x_1 - x_0}^{x_1 + x_0} \Big(|f^0|^2 + |f^2|^2 \Big) d\tau \right\}.
\end{aligned}
$$

Thus, the Cauchy problem (3.2.5) with $f^\alpha \in C^1(\mathbb{R}^1, \mathbb{C}^1), \ \alpha = 0, \ldots, 3$ has the unique solution.

In the case involved we have succeeded in integrating a nonlinear system of PDEs due to the fact that it is equivalent to the linear system (3.2.2). In some cases the nonlocal linearization method makes it possible to construct wide classes of exact solutions of essentially nonlinear PDEs. This is achieved by imposing such additional constraints on the equation under study that the system obtained is linearizable. A peculiar example is the generalized Thirring model

$$
\begin{aligned}
iu_y &= mv + \lambda_1 |v|^2 u, \\
iv_x &= mu + \lambda_2 |u|^2 v,
\end{aligned}
\tag{3.2.6}
$$

where $u = u(x, y), \ v = v(x, y)$ are complex-valued functions, $m, \ \lambda_1, \ \lambda_2$ are real constants.

Provided $\lambda_1 = \lambda_2 = \lambda$, system of PDEs (3.2.6) coincides with the classical Thirring model that is integrable by means of the inverse scattering method [249, 277]. As established by David [66] the generalized Thirring model (3.2.6) is also integrable by the mentioned method and, therefore, has soliton solutions.

Here we restrict ourselves to the case

$$
\lambda_1 = \lambda, \quad \lambda_2 = -\lambda
$$

and consider the following system:

$$
\begin{aligned}
iu_y &= mv + \lambda |v|^2 u, \\
iv_x &= mu - \lambda |u|^2 v.
\end{aligned}
\tag{3.2.7}
$$

We will show that there exists a map of the set of solutions of the linear Klein-Gordon equation

$$
w_{xy} + m^2 w = 0 \tag{3.2.8}
$$

into the set of solutions of system of PDEs (3.2.7).

To this end we apply the following Ansatz [299]:

$$
\begin{aligned}
u &= F_1 \exp\{iG + (i\pi/4)\},\\
v &= F_2 \exp\{iG - (i\pi/4)\}
\end{aligned}
\tag{3.2.9}
$$

where $F_1,\ F_2,\ G$ are some real-valued functions.

Substitution of (3.2.9) into (3.2.7) yields an over-determined system of four nonlinear PDEs for $F_1,\ F_2,\ G$

$$
\begin{aligned}
&F_{1y} = -mF_2, \quad F_{2x} = mF_1,\\
&G_x = \lambda F_1^2, \quad G_y = -\lambda F_2^2.
\end{aligned}
$$

Since

$$
(G_x)_y = 2\lambda F_1 F_{1y} = -2\lambda m F_1 F_2 = -2\lambda F_2 F_{2x} = (G_y)_x,
$$

the above system is compatible and its general solution can be represented in the form

$$
\begin{aligned}
&F_1 = w(x,y), \quad F_2 = -m^{-1} w_y(x,y),\\
&G = \lambda \int\limits_A^x w^2(\tau,y)d\tau - \lambda m^{-2} \int\limits_B^y w_y^2(A,\tau)d\tau,
\end{aligned}
$$

where $A,\ B$ are some real constants and $w(x,y)$ is an arbitrary solution of (3.2.8).

Thus, each solution of the linear Klein-Gordon equation (3.2.8) gives rise to the exact solution of the nonlinear system (3.2.7) of the form

$$
\begin{aligned}
u &= w \exp\left\{(i\pi/4) + i\lambda \int\limits_A^x w^2(\tau,y)d\tau - i\lambda m^{-2} \int\limits_B^y w_y^2(A,\tau)d\tau\right\},\\
v &= -m^{-1} w_y \exp\left\{(-i\pi/4) + i\lambda \int\limits_A^x w^2(\tau,y)d\tau - i\lambda m^{-2} \int\limits_B^y w_y^2(A,\tau)d\tau\right\}.
\end{aligned}
$$

Due to invariance of system (3.2.7) under the one-parameter group of gauge transformations

$$
u' = u \exp\{i\theta\},\ v' = v \exp\{i\theta\},\ \theta \in \mathbb{R}^1
$$

the solution obtained can be rewritten in the following equivalent form:

$$
\begin{aligned}
u &= w \exp\left\{ i\lambda \int\limits_A^x w^2(\tau, y) d\tau - i\lambda m^{-2} \int\limits_B^y w_y^2(A, \tau) d\tau \right\}, \\
v &= i m^{-1} w_y \exp\left\{ i\lambda \int\limits_A^x w^2(\tau, y) d\tau - i\lambda m^{-2} \int\limits_B^y w_y^2(A, \tau) d\tau \right\}.
\end{aligned}
\tag{3.2.10}
$$

The above formulae can be interpreted as a linearizing nonlocal transformation, since functions (3.2.10) satisfy system of PDEs (3.2.7) iff the function $w(x, y)$ satisfies the linear Klein-Gordon equation (3.2.8). However in this way only a part of solutions of system under study is obtained. Therefore, we can speak about partial linearization of the generalized Thirring model (another example of partial linearization is considered in Section 2.8).

3.3. Two-dimensional classical electrodynamics equations

The change of variables (3.2.1) proves to be efficient when constructing the general solution of the system of nonlinear PDEs (3.1.2).

Writing the first equation (3.1.2) component-wise and passing to the cone variables ξ, η we come to the following system of PDEs for the functions $\psi^0(\xi, \eta), \dots, \psi^3(\xi, \eta)$:

$$
\begin{aligned}
i\partial_\eta \psi^0 &= (e/2)(\widetilde{A}_0 + \widetilde{A}_1)\psi^0, \\
i\partial_\xi \psi^1 &= (e/2)(\widetilde{A}_0 - \widetilde{A}_1)\psi^1, \\
i\partial_\eta \psi^2 &= (e/2)(\widetilde{A}_0 + \widetilde{A}_1)\psi^2, \\
i\partial_\xi \psi^3 &= (e/2)(\widetilde{A}_0 - \widetilde{A}_1)\psi^3,
\end{aligned}
\tag{3.3.1}
$$

where

$$
\widetilde{A}_\mu = A_\mu\Big((1/2)(\xi + \eta),\, (1/2)(\xi - \eta)\Big).
$$

On making the change of variables

$$
\begin{aligned}
\psi^0 &= \varphi^0(\xi, \eta) \exp\left\{ -(ie/2) \int (\widetilde{A}_0 + \widetilde{A}_1) d\eta \right\}, \\
\psi^1 &= \varphi^1(\xi, \eta) \exp\left\{ -(ie/2) \int (\widetilde{A}_0 - \widetilde{A}_1) d\xi \right\},
\end{aligned}
$$

$$\begin{aligned}
\psi^2 &= \varphi^2(\xi,\eta)\exp\Big\{-(ie/2)\int(\widetilde{A}_0+\widetilde{A}_1)d\eta\Big\}, \\
\psi^3 &= \varphi^3(\xi,\eta)\exp\Big\{-(ie/2)\int(\widetilde{A}_0-\widetilde{A}_1)d\xi\Big\},
\end{aligned} \tag{3.3.2}$$

we rewrite (3.3.1) in the following way:

$$\begin{aligned}
&\partial_\eta\varphi^0=0, \quad \partial_\xi\varphi^1=0,\\
&\partial_\eta\varphi^2=0, \quad \partial_\xi\varphi^3=0.
\end{aligned}$$

The general solution of the above system is given by formulae (3.2.3). Consequently, the general solution of equations (3.3.1) is of the form

$$\begin{aligned}
\psi^0 &= U^0(\xi)\exp\Big\{-(ie/2)\int(\widetilde{A}_0+\widetilde{A}_1)d\eta\Big\}, \\
\psi^1 &= U^1(\eta)\exp\Big\{-(ie/2)\int(\widetilde{A}_0-\widetilde{A}_1)d\xi\Big\}, \\
\psi^2 &= U^2(\xi)\exp\Big\{-(ie/2)\int(\widetilde{A}_0+\widetilde{A}_1)d\eta\Big\}, \\
\psi^3 &= U^3(\eta)\exp\Big\{-(ie/2)\int(\widetilde{A}_0-\widetilde{A}_1)d\xi\Big\},
\end{aligned} \tag{3.3.3}$$

where $U^\mu \in C^1(\mathbb{R}^1,\mathbb{C}^1),\ \mu=0,\ldots,3$ are arbitrary functions.

Substituting expressions (3.3.3) into the remaining equations of system (3.1.2) we get an over-determined system of PDEs for $A_0,\ A_1$

$$\begin{aligned}
\partial_1(\partial_1A_0+\partial_0A_1) &= e\Big(|U^0|^2+|U^1|^2+|U^2|^2+|U^3|^2\Big),\\
\partial_0(\partial_1A_0+\partial_0A_1) &= e\Big(|U^0|^2-|U^1|^2+|U^2|^2-|U^3|^2\Big).
\end{aligned} \tag{3.3.4}$$

Introducing the new dependent variable

$$w=\partial_1A_0+\partial_0A_1$$

we rewrite equations (3.3.4) as follows

$$\partial_\xi w = e\Big(|U^0(\xi)|^2+|U^2(\xi)|^2\Big), \quad \partial_\eta w = e\Big(|U^1(\eta)|^2+|U^3(\eta)|^2\Big),$$

whence

$$w = e\int^{\xi}\Big(|U^0(z)|^2 + |U^2(z)|^2\Big)dz + e\int^{\eta}\Big(|U^1(z)|^2 + |U^3(z)|^2\Big)dz.$$

Consequently, to determine $A_0(x_0,x_1)$, $A_1(x_0,x_1)$ it is necessary to integrate PDE

$$\begin{aligned} A_{0x_1} + A_{1x_0} &= e\int^{x_0+x_1}\Big(|U^0(z)|^2 + |U^2(z)|^2\Big)dz \\ &\quad + e\int^{x_0-x_1}\Big(|U^1(z)|^2 + |U^3(z)|^2\Big)dz, \end{aligned}$$

whose general solution is of the form

$$\begin{aligned} A_0 &= e\int^{x_0+x_1}\int^{z}\Big(|U^0(\xi)|^2 + |U^2(\xi)|^2\Big)d\xi dz + \partial_0 f, \\ A_1 &= e\int^{x_0-x_1}\int^{z}\Big(|U^1(\eta)|^2 + |U^3(\eta)|^2\Big)d\eta dz - \partial_1 f \end{aligned} \tag{3.3.5}$$

with an arbitrary function $f = f(x_0,x_1) \in C^3(\mathbb{R}^2,\mathbb{R}^1)$.

Substitution of (3.3.5) into (3.3.3) gives rise to the final expressions for the functions $\psi^0(x),\ldots,\psi^3(x)$

$$\begin{aligned} \begin{Bmatrix}\psi^0(x)\\ \psi^2(x)\end{Bmatrix} &= \begin{Bmatrix}U^0(x_0+x_1)\\ U^2(x_0+x_1)\end{Bmatrix}\exp\Bigg\{-ief + (ie^2/2) \\ &\quad\times\Bigg((x_1-x_0)\int^{x_0+x_1}\int^{z}\Big(|U^0(\xi)|^2 + |U^2(\xi)|^2\Big)d\xi dz \\ &\quad - \int^{x_0-x_1}\int^{z_2}\int^{z_1}\Big(|U^1(\eta)|^2 + |U^3(\eta)|^2\Big)d\eta dz_1 dz_2\Bigg)\Bigg\}, \\ \begin{Bmatrix}\psi^1(x)\\ \psi^3(x)\end{Bmatrix} &= \begin{Bmatrix}U^1(x_0-x_1)\\ U^3(x_0-x_1)\end{Bmatrix}\exp\Bigg\{-ief + (ie^2/2) \\ &\quad\times\Bigg((x_1+x_0)\int^{x_0-x_1}\int^{z}\Big(|U^1(\eta)|^2 + |U^3(\eta)|^2\Big)d\eta dz \\ &\quad - \int^{x_0+x_1}\int^{z_2}\int^{z_1}\Big(|U^0(\xi)|^2 + |U^2(\xi)|^2\Big)d\xi dz_1 dz_2\Bigg)\Bigg\}. \end{aligned} \tag{3.3.6}$$

Choosing arbitrary functions $U^\mu,\ \mu = 0, \dots, 3$ in proper way we can obtain special classes of solutions possessing some additional properties.

If, for example, we choose in formulae (3.3.3), (3.3.6)

$$\begin{aligned} &U^\mu(z) = C^\mu \left(\frac{d}{dz} \exp\{-z^2\} \right)^{1/2}, \ \mu = 0, 2, \\ &U^1(z) = U^3(z) = 0, \ f = 0, \end{aligned} \tag{3.3.7}$$

where $C^0,\ C^2$ are complex constants, then the corresponding wave function $\psi(x)$ is localized in the neighborhood of the point $x_0 = x_1$. Consequently, solution (3.3.6), (3.3.7) is a solitary wave propagating with the velocity $v = 1$.

Substitution of expressions (3.3.7) into (3.3.5) yields the following formulae for $A_0(x),\ A_1(x)$:

$$\begin{aligned} A_0(x) &= e\Big(|C^0|^2 + |C^2|^2\Big) \int^{x_0 + x_1} \exp\{-\tau^2\} d\tau, \\ A_1(x) &= 0, \end{aligned}$$

whence it follows that the electro-magnetic field

$$F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$$

is localized in the neighborhood of the point $x_1 = x_0$ and vanishes rapidly as $|x_1| \to +\infty$.

Thus, we can interpret (3.3.6), (3.3.7) as a wave function of a particle moving in the electro-magnetic field $F_{\mu\nu}$, which is localized in the neighborhood of the line $x_1 = x_0$.

In conclusion, we note that the method described above has been used in [293] to construct the general solution of the following two-dimensional system of nonlinear PDEs:

$$\begin{aligned} &\Big(i\gamma_\mu \partial_\mu - e\gamma_\mu A^\mu - \lambda\gamma_\mu(\bar\psi\gamma^\mu\psi)\Big)\psi = 0, \\ &\partial_\nu \partial^\nu A_\mu - \partial^\mu \partial_\nu A_\nu = -e\bar\psi\gamma_\mu\psi, \quad \mu, \nu = 0, 1, \end{aligned}$$

which can be represented in the form

$$\begin{aligned} A_\mu(x) &= \widetilde{A}_\mu(x), \\ \left\{ \begin{matrix} \psi^0(x) \\ \psi^2(x) \end{matrix} \right\} &= \left\{ \begin{matrix} \tilde\psi^0(x) \\ \tilde\psi^2(x) \end{matrix} \right\} \exp\left\{ -i\lambda \int^{x_0 - x_1} \Big(|U^1(\eta)|^2 + |U^3(\eta)|^2\Big) d\eta \right\}, \\ \left\{ \begin{matrix} \psi^1(x) \\ \psi^3(x) \end{matrix} \right\} &= \left\{ \begin{matrix} \tilde\psi^1(x) \\ \tilde\psi^3(x) \end{matrix} \right\} \exp\left\{ -i\lambda \int^{x_0 + x_1} \Big(|U^0(\xi)|^2 + |U^2(\xi)|^2\Big) d\xi \right\}, \end{aligned}$$

functions $\widetilde{A}_0(x)$, $\widetilde{A}_1(x)$, $\tilde{\psi}^0(x), \ldots, \tilde{\psi}^3(x)$ being given by formulae (3.3.5), (3.3.6) correspondingly.

3.4. General solutions of Galilei-invariant spinor equations

Let us rewrite system (3.1.3) in the equivalent form by introducing new functions $F(x)$, $f(x)$

$$\begin{aligned} \psi_{x_0} &= -F, \\ \psi_{x_1} &= i\lambda f_{x_1}\gamma_1\psi - \gamma_1(\gamma_0+\gamma_4)F, \\ f_{x_1} &= (\psi^\dagger\psi + \bar{\psi}\gamma_4\psi)^{1/2k}. \end{aligned} \tag{3.4.1}$$

Consider now the second equation of system (3.4.1) as a system of ODEs with respect to x_1. Since this system is linear, it is possible to apply the standard method of variation of an arbitrary constant [197]. The general solution of the homogeneous part of the system in question is given by the formula

$$\psi(x) = \exp\{i\lambda\gamma_1 f(x)\}\varphi(x_0),$$

where $\varphi(x_0)$ is an arbitrary four-component function. Consequently, the general solution of the second equation from system (3.4.1) has the form

$$\begin{aligned} \psi(x) \;=\;& \exp\{i\lambda\gamma_1 f(x)\}\Bigg(\varphi(x_0) + \gamma_1\int^{x_1}\exp\{-i\lambda\gamma_1 f(x_0,z)\} \\ & \times(\gamma_0+\gamma_4)F(x_0,z)dz\Bigg) \end{aligned} \tag{3.4.2}$$

Multiplying both parts of the first equation from (3.4.1) by the matrix $\gamma_0+\gamma_4$ we have

$$(\gamma_0+\gamma_4)F = -(\gamma_0+\gamma_4)\psi_{x_0},$$

whence

$$(\gamma_0+\gamma_4)F = -\exp\{-i\lambda\gamma_1 f\}(\gamma_0+\gamma_4)(\dot{\varphi} + i\lambda\gamma_1 f_{x_0}\varphi).$$

Consequently, formula (3.4.2) takes the form

$$\begin{aligned} \psi(x) \;=\;& \exp\{i\lambda\gamma_1 f(x)\}\Bigg(\varphi(x_0) + \gamma_1(\gamma_0+\gamma_4)\int^{x_1}\exp\{2i\lambda\gamma_1 \\ & \times f(x_0,z)\}\Big(\dot{\varphi}(x_0) + i\lambda\gamma_1 f_{x_0}(x_0,z)\varphi(x_0)\Big)dz\Bigg). \end{aligned} \tag{3.4.3}$$

Substitution of the above expression into the third equation of system (3.4.1) yields the nonlinear ODE for a function $f(x)$

$$f_{x_1} = (A_1 \cosh 2\lambda f + A_2 \sinh 2\lambda f)^{1/2k}, \qquad (3.4.4)$$

where

$$A_1 = \bar{\varphi}(\gamma_0 + \gamma_4)\varphi, \quad A_2 = i\bar{\varphi}(\gamma_0 + \gamma_4)\gamma_1\varphi.$$

The general solution of (3.4.4) is given by the quadrature

$$\int^{f(x_0, x_1)} (A_1 \cosh 2\lambda z + A_2 \sinh 2\lambda z)^{-1/2k} dz = x_1 + C(x_0), \qquad (3.4.5)$$

$C(x_0)$ being an arbitrary smooth real-valued function.

Thus, the general solution of nonlinear system of PDEs (3.1.3) has the form (3.4.3), function $f = f(x_0, x_1)$ being determined by implicit formula (3.4.5).

Using formulae (3.4.3), (3.4.5) it is not difficult to obtain the general solution of the linear equation

$$\Big(i(\gamma_0 + \gamma_4)\partial_0 + i\gamma_1\partial_1 - \lambda\Big)\psi(x) = 0,$$

which is of the form

1) under $\lambda \neq 0$

$$\psi(x) = \exp\{i\lambda\gamma_1 x_1\}\Big(\varphi(x_0) + (i/2\lambda)(\gamma_0 + \gamma_4)\exp\{2i\lambda\gamma_1 x_1\}\dot{\varphi}(x_0)\Big),$$

2) under $\lambda = 0$

$$\psi(x) = \varphi(x_0) + x_1\gamma_1(\gamma_0 + \gamma_4)\dot{\varphi}(x_0).$$

Reduction of the nonlinear Dirac equation (2.4.1) by means of the Ansatz

$$\begin{aligned} &\psi(x) = \varphi(\xi, \omega), \\ &\xi = x_0 + x_3, \ \omega = x_2 \end{aligned} \qquad (3.4.6)$$

invariant under the two-parameter group with generators $\partial_0 - \partial_3, \ \partial_1$ yields the two-dimensional system of PDEs

$$\Big(i(\gamma_0 + \gamma_3)\partial_\xi + i\gamma_2\partial_2 - \lambda(\bar{\psi}\psi)^r\Big)\psi = 0, \ r = 1/2k, \qquad (3.4.7)$$

which can also be integrated by means of the above described trick [152, 304].

Rewriting system of PDEs (3.4.7) in the equivalent form (3.4.1) we have

$$\begin{aligned} \varphi_\xi &= F, && (3.4.8)\\ \varphi_\omega &= if_\omega\gamma_2\varphi + \gamma_2(\gamma_0+\gamma_3)F, && (3.4.9)\\ f_\omega &= \lambda(\bar\psi\psi)^r. && (3.4.10) \end{aligned}$$

Integration of system (3.4.9) by the method of variation of an arbitrary constant with respect to ω yields the following expression for φ:

$$\begin{aligned} \varphi(\xi,\omega) &= \exp\{i\gamma_2 f\}\biggl(\Theta(\xi) + \gamma_2(\gamma_0+\gamma_3)\int\limits_0^\omega \exp\{i\gamma_2 f(\xi,z)\}\\ &\quad \times F(\xi,z)dz\biggr), \end{aligned}$$

where $\Theta(\xi)$ is an arbitrary four-component function-column.

As due to (3.4.8) the equation

$$(\gamma_0+\gamma_3)\varphi_\xi = (\gamma_0+\gamma_3)F$$

holds, we can exclude from the above equality the function F

$$\begin{aligned} \varphi(\xi,\omega) &= \exp\{i\gamma_2 f\}\biggl(\Theta + \gamma_2(\gamma_0+\gamma_3)\int\limits_0^\omega \exp\{2i\gamma_2 f(\xi,z)\}\\ &\quad \times\Bigl(\dot\Theta + if_\xi(\xi,z)\gamma_2\Theta\Bigr)dz\biggr). \end{aligned} \qquad (3.4.11)$$

The only thing left is to substitute (3.4.11) into (3.4.10). As a result, we get an integro-differential equation for $f = f(\xi,\omega)$

$$f_\omega = \lambda\biggl(A + B\int\limits_0^\omega \cosh 2f dz + C\int\limits_0^\omega \sinh 2f dz\biggr)^r, \qquad (3.4.12)$$

where

$$\begin{aligned} &A = \bar\Theta\Theta,\\ &B = \bar\Theta\gamma_2(\gamma_0+\gamma_3)\dot\Theta - \dot{\bar\Theta}\gamma_2(\gamma_0+\gamma_3)\Theta,\\ &C = i\Bigl(\bar\Theta(\gamma_0+\gamma_3)\dot\Theta - \dot{\bar\Theta}(\gamma_0+\gamma_3)\Theta\Bigr). \end{aligned}$$

The general solution of equation (3.4.12) has been constructed in [304]. Since its explicit form depends on relations between B and C, we have to consider four inequivalent cases.

Case 1. $B = \pm C,\ B \neq 0$

a) $r \neq -1$

$$f = \pm(1/2)\ln\Big(\varepsilon \pm 2\lambda B^{-1}(r+1)^{-1}(A+Bg)^{r+1}\Big),$$
$$\int\limits_0^{g(\xi,\omega)} \Big[\varepsilon \pm 2\lambda B^{-1}(r+1)^{-1}(A+B\tau)^{r+1}\Big]^{-1} d\tau = \omega;$$

b) $r = 1$

$$f = \pm(1/2)\ln\Big(\varepsilon \pm 2\lambda B^{-1}\ln(A+Bg)\Big),$$
$$\int\limits_0^{g(\xi,\omega)} \Big[\varepsilon \pm 2\lambda B^{-1}\ln(A+B\tau)\Big]^{-1} d\tau = \omega.$$

Case 2. $B^2 > C^2 \Leftrightarrow B = \alpha(\xi)\cosh 2\beta(\xi),\ B = \alpha(\xi)\sinh 2\beta(\xi)$

a) $r \neq -1$

$$\cosh 2(f+\beta) = \Big[1 + \Big(\varepsilon + 2\lambda\alpha^{-1}(r+1)^{-1}(A+\alpha g)^{r+1}\Big)^2\Big]^{1/2},$$
$$\int\limits_0^{g(\xi,\omega)} \Big[1 + \Big(\varepsilon + 2\lambda\alpha^{-1}(r+1)^{-1}(A+\alpha\tau)^{r+1}\Big)^2\Big]^{-1/2} d\tau = \omega;$$

b) $r = -1$

$$\cosh 2(f+\beta) = \Big[1 + \Big(\varepsilon + 2\lambda\alpha^{-1}\ln(A+\alpha g)\Big)^2\Big]^{1/2},$$
$$\int\limits_0^{g(\xi,\omega)} \Big[1 + \Big(\varepsilon + 2\lambda\alpha^{-1}\ln(A+\alpha\tau)\Big)^2\Big]^{-1/2} d\tau = \omega.$$

Case 3. $B^2 < C^2 \Leftrightarrow B = \alpha(\xi)\sinh 2\beta(\xi),\ B = \alpha(\xi)\cosh 2\beta(\xi)$

a) $r \neq -1$

$$\sinh 2(f+\beta) = \Big[-1 + \Big(\varepsilon + 2\lambda\alpha^{-1}(r+1)^{-1}(A+\alpha g)^{r+1}\Big)^2\Big]^{1/2},$$

$$\int\limits_0^{g(\xi,\omega)} \Big[-1 + \Big(\varepsilon + 2\lambda\alpha^{-1}(r+1)^{-1}(A+\alpha\tau)^{r+1}\Big)^2\Big]^{-1/2} d\tau = \omega.$$

b) $r = -1$

$$\sinh 2(f+\beta) = \Big[-1 + \Big(\varepsilon + 2\lambda\alpha^{-1}\ln(A+\alpha g)\Big)^2\Big]^{1/2},$$

$$\int\limits_0^{g(\xi,\omega)} \Big[-1 + \Big(\varepsilon + 2\lambda\alpha^{-1}\ln(A+\alpha\tau)\Big)^2\Big]^{-1/2} d\tau = \omega.$$

Case 4. $B = C = 0$

$$f = \lambda A^r \omega.$$

In the above formulae parameter ε takes the values $-1,\ 0,\ 1$.

Thus, we have constructed the general solution of system (3.4.7). Substitution of the obtained expression for the four-component function $\varphi = \varphi(\xi,\omega)$ into Ansatz (3.4.6) with $r = 1/2k$ yields a class of exact solutions of the nonlinear Dirac equation (2.4.1). And what is more, this class contains four arbitrary complex functions of $\xi = x_0 + x_3$ (components of the function $\Theta(\xi)$). Such arbitrariness enables us to solve a wide class of Cauchy problems for the system of nonlinear PDEs (2.4.1).

NONLINEAR GALILEI-INVARIANT SPINOR EQUATIONS

In the present chapter we investigate linear and nonlinear systems of PDEs for the spinor field admitting the Galilei group $G(1,3)$. Wide classes of nonlinear first-order spinor PDEs invariant under the group $G(1,3)$ and its extensions, groups $G_1(1,3)$ and $G_2(1,3)$, are described. All Ansätze for the spinor field $\psi(t,\,\vec{x})$ invariant under the $G(1,3)$ non-conjugate three-parameter subgroups of the Galilei group are obtained. With the use of these Ansätze the multi-parameter families of exact solutions of a nonlinear Galilei-invariant spinor equation are constructed. In addition, we briefly consider the second-order spinor PDEs invariant under the group $G(1,3)$.

4.1. Nonlinear equations for the spinor field invariant under the group G(1,3) and its extensions

In spite of the fact that the Galilei relativity principle is known for more than 300 years, the concept of the Galilei group has arisen only recently (1950–1970). It is even more surprising, if we take into account that Sophus Lie has discovered this group as early as in 1889. It was Lie who established that the one-dimensional linear heat-transfer equation (which up to the constant factor coincided with the Schrödinger equation) was invariant with respect to the translation group, Galilei transformation, scale and projective transformations. Simultaneously, he discovered a projective representation of the Galilei group $G(1,1)$.

Bargmann and Wigner [18, 191] have rediscovered projective representa-

tions of the Galilei group and showed the fundamental role played by these in the quantum theory. Since Bargmann's and Wigner's works the Galilei group is intensively used by specialists dealing with mathematical physics problems.

A Galilei-invariant equation for a particle with the spin $s = 1/2$ was suggested in [172, 212]. Systematic study of the first-order equations invariant under the group $G(1,3)$ was begun by Lévy-Leblond [212, 213] and Hagen, Hurley [178, 186]. The algebraic-theoretical derivation and detailed investigation of the new classes of linear Galilei-invariant equations for particles with arbitrary spins were carried out in [114]–[116], [118, 119, 130]. Some nonlinear Galilei-invariant systems of PDEs were considered in [130, 259, 296].

A Galilei-invariant equation for a particle with the spin $s = 1/2$ can be represented in the form [296]

$$\{-i(\gamma_0 + \gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0 - \gamma_4)\}\psi(t, \vec{x}) = 0, \tag{4.1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial_{x_a}$, $a = 1, 2, 3$, $m = \text{const}$, $\psi = \psi(t, \vec{x})$ is a four-component complex-valued function (spinor), $\vec{x} \in \mathbb{R}^3$, $t \in \mathbb{R}^1$.

In the process of derivation of equation (4.1.1) the Dirac's heuristic trick was used. Namely, one looked for a first-order system of PDEs with constant matrix coefficients for a spinor $\psi(t, \vec{x})$ whose components satisfied the Schrödinger equation

$$(4im\partial_t - \partial_a\partial_a)\psi^\alpha(t, \vec{x}) = 0, \quad \alpha = 0, \ldots, 3, \tag{4.1.2}$$

whence it immediately followed that up to equivalence the equation required had the form (4.1.1) (to obtain (4.1.2) one has to act with the operator $-i(\gamma_0 + \gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0 - \gamma_4))$ on system (4.1.1). Let us note that the more traditional notation of the equation for a Galilean particle with the spin $s = 1/2$

$$\{i(1 + \gamma_0)\partial_t + i\gamma_a\partial_a + m(1 - \gamma_0)\}\,\psi(t, \vec{x}) = 0$$

is obtained if we multiply (4.1.1) by the matrix γ_4 and change the dependent variable $\psi \to \psi' = 2^{-1/2}(1 + \gamma_4)\psi$.

1. Local symmetry of system of PDEs (4.1.1). The Lie symmetry of PDE (4.1.1) for $m \neq 0$ is well-known. In particular, in [119, 130, 259] it was established that (4.1.1) is invariant under the 13-parameter generalized Galilei group $G_2(1,3)$ (it is also called the Schrödinger group and denoted $Sch(1,3)$). We will prove the assertions describing the maximal (in Lie sense) invariance group admitted by equation (4.1.1) for both cases $m \neq 0$ and $m = 0$.

Theorem 4.1.1. *The maximal local invariance group of equation (4.1.1) with $m \neq 0$ is the 14-parameter group $G^{(1)} = G_2(1,3) \otimes I(1)$,*[3] *where $G_2(1,3)$ is the 13-parameter generalized Galilei group having the generators*

$$\begin{aligned}
P_0 &= \partial_t, \quad P_a = \partial_a, \quad M = 2im, \\
J_{ab} &= x_a\partial_b - x_b\partial_a - (1/2)\gamma_a\gamma_b, \\
G_a &= t\partial_a + 2imx_a + (1/2)(\gamma_0 + \gamma_4)\gamma_a, \\
D &= 2t\partial_t + x_a\partial_a + 2 - (1/2)\gamma_0\gamma_4, \\
A &= tD - t^2\partial_t + imx_a x_a + (1/2)(\gamma_0 + \gamma_4)\gamma_a x_a
\end{aligned} \tag{4.1.3}$$

and $I(1)$ is the following one-parameter group

$$x'_\mu = x_\mu, \quad \psi'(t', \vec{x}') = e^\theta \psi(t, \vec{x}).$$

In the above formulae $a, b = 1, 2, 3,\ a \neq b$, $\theta = \text{const}$ is a group parameter.

The proof is carried out by means of the Lie method. According to [236] the operator

$$\begin{aligned}
Q = {}& \xi_0(x, \psi^*, \psi)\partial_t + \xi_a(x, \psi^*, \psi)\partial_a \\
&+ \eta^\alpha(x, \psi^*, \psi)\partial_{\psi^\alpha} + \eta^{*\alpha}(x, \psi^*, \psi)\partial_{\psi^{*\alpha}}
\end{aligned} \tag{4.1.4}$$

generates an invariance group of PDE (4.1.1) iff the following relations hold

$$\begin{aligned}
&\widetilde{Q}\{-i(\gamma_0 + \gamma_4)\psi_t + i\gamma_a\psi_{x_a} + m(\gamma_0 - \gamma_4)\psi\}\Big|_{[L]} = 0, \\
&\widetilde{Q}\{i(\gamma_0^* + \gamma_4^*)\psi_t^* - i\gamma_a^*\psi_{x_a}^* + m(\gamma_0^* - \gamma_4^*)\psi^*\}\Big|_{[L]} = 0,
\end{aligned} \tag{4.1.5}$$

where $\widetilde{Q}$ is the first prolongation of the operator Q. By the symbol $[L]$ we designate the set of solutions of equation (4.1.1).

Relations (4.1.5) yield the following determining equations for the coefficients of the operator Q:

$$\begin{aligned}
&\eta = -(\tilde{a} + b_\mu\gamma^\mu + c_{\mu\nu}\gamma^\mu\gamma^\nu + d_\mu\gamma^\mu\gamma_4 + e)\psi + \Omega\psi^* + \Psi, \\
&\eta^* = -(\tilde{a}^* + b_\mu^*\gamma^{\mu*} + c_{\mu\nu}^*\gamma^{\mu*}\gamma^{\nu*} + d_\mu^*\gamma^{\mu*}\gamma_4^* + e^*)\psi^* + \Omega^*\psi + \Psi^*,
\end{aligned}$$

[3] Since equation (4.1.1) is linear, it admits an infinite-parameter group $\psi' = \psi + \theta\Psi(t, \vec{x})$, where θ is a group parameter and Ψ is an arbitrary solution of the system of PDEs (4.1.1). Such a symmetry gives no essential information about the structure of the solutions of the equation under consideration and therefore is neglected.

$$\begin{aligned}
&\partial_t\xi_0 + 2d_0 = \partial_1\xi_1 = \partial_2\xi_2 = \partial_3\xi_3, \quad \partial_a\xi_0 = 0,\\
&\partial_t\xi_a = 2d_a = -4c_{0a}, \quad \partial_b\xi_a = -\partial_a\xi_b = 4c_{ab},\\
&\partial_a e = \partial_t\partial_c\xi_b, \quad (a,b,c) = \text{ cycle } (1,2,3), \quad b_a = 0,\\
&m\, e = 0, \quad m(\partial_t\xi_0 + 4d_0) = 0, \quad \partial_t\Big(\tilde{a} - d_0 - (1/2)\partial_a\xi_a\Big) = 0,\\
&\partial_b\tilde{a} - (1/2)\partial_a\partial_a\xi_b - 2im\partial_t\xi_b = 0, \quad \partial_a d_0 = 0,\\
&\partial_t\Omega = \partial_a\Omega = 0, \quad (\gamma_0+\gamma_4)\Omega = -\Theta(\gamma_0-\gamma_4),\\
&\gamma_a\Omega = -\Omega\gamma_a^*, \quad m(\gamma_0-\gamma_4)\Omega = m\Theta(\gamma_0+\gamma_4),
\end{aligned} \tag{4.1.6}$$

where η is the four-component function $\{\eta^0, \eta^1, \eta^2, \eta^3\}^T$, Ψ is an arbitrary solution of the system of PDEs (4.1.1), $\Omega,\ \Theta$ are complex (4×4)-matrices, indices $a,\ b,\ c$ take the values 1, 2, 3 and what is more $a \neq b$.

Since $m \neq 0$, from (4.1.6) it follows that $e = 0$, $f = -2d_0$ and besides $d_0 = d_0(t)$. Due to this fact the equations for functions $\xi_0,\ \xi_1,\ \xi_2,\ \xi_3$ are rewritten in the form

$$\begin{aligned}
&\partial_a\xi_0 = 0, \quad \partial_t\xi_0 = -4d_0(t), \quad \partial_b\xi_a = -\partial_a\xi_b, \quad a \neq b,\\
&\partial_1\xi_1 = \partial_2\xi_2 = \partial_3\xi_3 = -2d_0(t), \quad \partial_t\partial_a\xi_b = 0, \quad a \neq b,
\end{aligned}$$

whence it follows that $\partial_b\partial_a\xi_a = 0$ (no summation over a) and what is more the equalities

$$\partial_a\partial_b\xi_c = -\partial_a\partial_c\xi_b = -\partial_c\partial_a\xi_b = \partial_c\partial_b\xi_a = \partial_b\partial_c\xi_a = -\partial_b\partial_a\xi_c,$$

where $(a,b,c) = \text{cycle}(1,2,3)$, hold. Consequently, $\partial_a\partial_b\xi_c = -\partial_a\partial_b\xi_c = 0$ which implies that the functions ξ_μ are linear in the variables $x_1,\ x_2,\ x_3$. Due to this fact it is not difficult to integrate the system of PDEs (4.1.6). Its general solution under $m \neq 0$ has the form

$$\begin{aligned}
&\xi_0 = A_1t^2 + 2A_2t + A_3,\\
&\xi_a = B_{ab}x_b + (A_1t + A_2)x_a + C_at + D_a,\\
&\tilde{a} = imA_1x_ax_a + 2imC_ax_a + 2A_1t + 2A_2 + A_4,\\
&c_{ab} = (1/4)B_{ab}, \quad d_a = (1/2)(A_1x_a + C_a),\\
&c_{0a} = -(1/4)(A_1x_a + C_a), \quad d_0 = -(1/2)(A_1t + A_2),\\
&b_0 = b_a = e = 0, \quad \Omega = 0, \quad \Theta = 0,
\end{aligned}$$

where $A_1, \ldots, A_4,\ B_{ab},\ C_a,\ D_a$ are arbitrary real constants, $B_{ab} = -B_{ba}$, $a, b = 1, 2, 3$.

Substitution of the above results into (4.1.4) shows that the most general infinitesimal operator Q admitted by the equation under study is a linear combination of operators (4.1.3), $I = \psi^{\alpha}\partial_{\psi^{\alpha}} + \psi^{\alpha *}\partial_{\psi^{\alpha *}}$ and $Q' = \Psi^{\alpha}(t,\, \vec{x})\partial_{\psi^{\alpha}} + \Psi^{*\alpha}(t,\, \vec{x})\partial_{\psi^{*\alpha}}$. Consequently, operators (4.1.3) together with the operators $I,\ Q'$ form the basis of the maximal invariance algebra admitted by the system of PDEs (4.1.1). The theorem is proved. $\triangleright$

Note 4.1.1. Operators $P_0,\ P_a,\ M,\ J_{ab},\ G_a$ form a basis of the Lie algebra of the Galilei group which is called the Galilei algebra $AG(1,3)$.

Note 4.1.2. Operators $P_0,\ P_a,\ M,\ J_{ab},\ G_a,\ D$ form a basis of the Lie algebra of the extended Galilei group $G_1(1,3)$ which is called the extended Galilei algebra $A\widetilde{G}_1(1,3)$.

Theorem 4.1.2. *The maximal local invariance group admitted by (4.1.1) with* $m = 0$ *is the infinite-parameter Lie group having the generators*[4]

$$\begin{aligned}
A_\infty &= \varphi_0(t)\partial_t + \dot\varphi_0(t)x_a\partial_a + (3/2)\dot\varphi_0(t) \\
&\quad +(1/2)\ddot\varphi_0(t)(\gamma_0+\gamma_4)\gamma_a x_a, \\
G_\infty &= \varphi_a(t)\partial_a + (1/2)(\gamma_0+\gamma_4)\gamma_a\dot\varphi_a(t), \\
D_\infty &= \varphi_4(t)\partial_t + (1/2)\dot\varphi_4(t)(1-\gamma_0\gamma_4), \\
T_\infty &= (\gamma_0+\gamma_4)\varphi_5(t), \\
J_\infty &= \varepsilon_{abc}\varphi_{5+a}(t)\Big(x_c\partial_b + (1/4)\gamma_b\gamma_c\Big) \\
&\quad +(1/2)(\gamma_0+\gamma_4)\gamma_a\dot\varphi_{5+a}(t)\gamma_b x_b, \\
M_1 &= \{C_1\psi\}^\alpha\partial_{\psi^\alpha} + \{C_1^*\psi^*\}^\alpha\partial_{\psi^{*\alpha}}, \\
M_2 &= \{C_2\gamma_2\gamma_4\psi^*\}^\alpha\partial_{\psi^\alpha} + \{C_2^*\gamma_2^*\gamma_4^*\psi\}^\alpha\partial_{\psi^{*\alpha}}, \\
M_3 &= \{C_3(\gamma_2+\gamma_3\gamma_1)\psi^*\}\partial_{\psi^\alpha} + \{C_3^*(\gamma_2^*+\gamma_3^*\gamma_1^*)\psi\}\partial_{\psi^{*\alpha}},
\end{aligned} \tag{4.1.7}$$

where $\varphi_0(t),\ \varphi_1(t),\ldots,\varphi_8(t)$ *are arbitrary smooth functions,* $\dot\varphi_s = d\varphi_s/dt$, $s = 0,\ldots,8$, *the symbol* $\{\psi\}^\alpha$ *denotes the* α-th *component of* ψ, $C_1,\ C_2,\ C_3$ *are arbitrary complex constants and*

$$\varepsilon_{abc} = \begin{cases} 1, & (a,b,c) = \mathrm{cycle}(1,2,3), \\ -1, & (a,b,c) = \mathrm{cycle}(2,1,3), \\ 0, & \textit{in the remaining cases.} \end{cases}$$

Proof. The determining equations for coefficients of the infinitesimal operator of the invariance group of equation (4.1.1) are of the form (4.1.6) under

[4] See the footnote on the page 217.

$m = 0$. The general solution of system of PDEs (4.1.6) with $m = 0$ is given by the following formulae:

$$\begin{aligned}
&\xi_0 = \varphi_0(t) - 2\varphi_4(t), \quad \xi_a = \varepsilon_{abc}x_b\varphi_{5+c}(t) + \dot{\varphi}(t)x_a + \varphi_a(t),\\
&\tilde{a} = (3/2)\dot{\varphi}_0(t) + C_1, \quad b_0 = e = \varphi_5(t) - (1/2)x_a\varphi_{5+a}(t),\\
&b_a = 0, \quad c_{ab} = (1/4)\varepsilon_{abc}\varphi_{5+c}(t),\\
&d_a = -2c_{0a} = -(1/2)\Big(\ddot{\varphi}_0(t)x_a + \dot{\varphi}_a(t) + \varepsilon_{abc}\dot{\varphi}_{5+c}(t)x_b\Big),\\
&\Omega = C_2\gamma_2\gamma_4 + C_3(\gamma_2 + \gamma_3\gamma_1),
\end{aligned}$$

where $\{C_1, C_2, C_3\} \subset \mathbb{C}^1$; $\varphi_0(t), \dots, \varphi_8(t)$ are arbitrary smooth functions. Substituting the above result into (4.1.4) we come to the conclusion that the most general infinitesimal operator admitted by equation (4.1.1) under $m = 0$ is a linear combination of operators (4.1.7) and $Q' = \Psi^\alpha(t, \vec{x})\partial_{\psi^\alpha} + \Psi^{*\alpha}(t, \vec{x})\partial_{\psi^{*\alpha}}$. Consequently, the operators listed in (4.1.7) together with the operator Q' form the basis of the maximal (in Lie sense) invariance algebra of (4.1.1). The theorem is proved. $\triangleright$

Note 4.1.3. The algebra (4.1.7) contains as a subalgebra the infinite-dimensional centerless Virasoro algebra with the following basis operators:

$$\begin{aligned}
q_n \equiv\ & A_\infty(t^n) = t^n\partial_t + nt^{n-1}x_a\partial_a + (3n/2)t^{n-1}\\
& +(1/2)n(n-1)t^{n-2}(\gamma_0 + \gamma_4)\gamma_a x_a,
\end{aligned}$$

which satisfy the commutation relations

$$[q_n,\, q_m] = (m-n)q_{n+m-1}, \quad n, m \in \mathbb{Z}.$$

The Virasoro algebra is a Kac-Moody-type algebra which plays an important role in the theory of two-dimensional dynamical systems (see, for example, [67, 276, 286]).

Note 4.1.4. On the set of solutions of system of PDEs (4.1.1) with $m = 0$ two inequivalent representations of the Lie algebra of the generalized Galilei group are realized

$$\begin{aligned}
1)\quad & P_0 = \partial_t, \quad P_a = \partial_a,\\
& G_a = t\partial_a + (1/2)(\gamma_0 + \gamma_4)\gamma_a,\\
& J_{ab} = x_a\partial_b - x_b\partial_a - (1/2)\gamma_a\gamma_b, \quad a \neq b,\\
& D = 2t\partial_t + x_a\partial_a + 2 - (1/2)\gamma_0\gamma_4,
\end{aligned}$$

$$A = tD - t^2\partial_t + (1/2)(\gamma_0 + \gamma_4)\gamma_a x_a;$$

$$\begin{aligned} 2)\quad & P_0 = \partial_t, \quad P_a = \partial_a, \\ & G_a = t\partial_a + (1/2)(\gamma_0 + \gamma_4)\gamma_a, \\ & J_{ab} = x_a\partial_b - x_b\partial_a - (1/2)\gamma_a\gamma_b, \quad a \neq b, \\ & D = t\partial_t + x_a\partial_a + 3/2, \\ & A = 2tD - t^2\partial_t + (\gamma_0 + \gamma_4)\gamma_a x_a. \end{aligned}$$

Further, we adduce transformation groups generated by the operators (4.1.3). To obtain a one-parameter transformation group generated by operator Q (4.1.4) it is necessary to solve the following Cauchy problem (the Lie equations):

$$\begin{aligned} & \frac{dt'}{d\tau} = \xi_0(t', \vec{x}', \psi'^*, \psi'), \quad \frac{dx'_a}{d\tau} = \xi_a(t', \vec{x}', \psi'^*, \psi'), \\ & \frac{d\psi'^\alpha}{d\tau} = \eta^\alpha(t', \vec{x}', \psi'^*, \psi'), \quad \frac{d\psi'^{*\alpha}}{d\tau} = \eta^{*\alpha}(t', \vec{x}', \psi'^*, \psi'), \\ & t'(0) = t, \quad x'_a(0) = x_a, \quad \psi'^\alpha(0) = \psi^\alpha, \quad \psi'^{*\alpha}(0) = \psi^{*\alpha}. \end{aligned} \tag{4.1.8}$$

Substituting into (4.1.8) functions ξ_μ, η^α, $\eta^{*\alpha}$ corresponding to the operators (4.1.3) and integrating the equations obtained we arrive at the following transformation groups:

$$P \;:\; \begin{cases} t' = t + \theta_0, \\ x'_a = x_a + \theta_a, \\ \psi'(t', \vec{x}') = \psi(t, \vec{x}); \end{cases} \tag{4.1.9}$$

$$J \;:\; \begin{cases} t' = t, \\ x'_a = \Big(\delta_{ab}\cos\theta + \varepsilon_{abc}\theta_c\theta^{-1}\sin\theta \\ \qquad + \theta_a\theta_b\theta^{-2}(1 - \cos\theta)\Big)x_b, \\ \psi'(t', \vec{x}') = \exp\{-(1/4)\varepsilon_{abc}\theta_a\gamma_b\gamma_c\}\psi(t, \vec{x}); \end{cases} \tag{4.1.10}$$

$$G \;:\; \begin{cases} t' = t, \\ x'_a = x_a + \theta_a t, \\ \psi'(t', \vec{x}') = \exp\Big\{-2im\Big(\theta_a x_a + (t/2)\theta_a\theta_a\Big) \\ \qquad - (1/2)(\gamma_0 + \gamma_4)\gamma_a\theta_a\Big\}\psi(t, \vec{x}); \end{cases} \tag{4.1.11}$$

$$D \;:\; \begin{cases} t' = te^{2\theta_0}, \\ x'_a = x_a e^{\theta_0}, \\ \psi'(t', \vec{x}') = \exp\{-2\theta_0 + (1/2)\theta_0\gamma_0\gamma_4\}\psi(t, \vec{x}); \end{cases} \tag{4.1.12}$$

$$A \quad : \quad \begin{cases} t' = t(1-\theta_0 t)^{-1}, \\ x'_a = x_a(1-\theta_0 t)^{-1}, \\ \psi'(t',\vec{x}') = (1-\theta_0 t)^2 \exp\Big\{-im\theta_0(1-\theta_0 t)^{-1}x_a x_a \\ \quad -(1/2t)\ln(1-\theta_0 t)\Big(t\gamma_0\gamma_4 + (\gamma_0+\gamma_4)\gamma_a x_a\Big)\Big\}\psi(t,\vec{x}); \end{cases} \tag{4.1.13}$$

$$M \quad : \quad \begin{cases} t' = t, \\ x'_a = x_a, \\ \psi'(t',\vec{x}') = e^{-2im\theta_0}\psi(t,\vec{x}); \end{cases} \tag{4.1.14}$$

where $P = \theta_0 P_0 + \theta_a P_a$, $J = (1/2)\varepsilon_{abc}\theta_a J_{bc}$, $G = \theta_a G_a$, θ_0, θ_a are group parameters, $\theta = (\theta_a\theta_a)^{1/2}$.

One can check by a direct computation that equation (4.1.1) is invariant under groups (4.1.9)–(4.1.14).

Note 4.1.5. Transformation groups corresponding to the operators (4.1.7) are given in [160].

2. Non-Lie symmetry of system of PDEs (4.1.1). As earlier (see Section 1.1) we designate by $\mathcal{M}_1$ the class of the first-order differential operators with complex matrix coefficients

$$X_0 = A_0(t,\vec{x})\partial_t + A_b(t,\vec{x})\partial_b + B(t,\vec{x})$$

acting on the space of four-component complex-valued functions $\psi = \psi(t,\vec{x})$. Below we adduce the assertions describing the symmetry of equation (4.1.1) in the class $\mathcal{M}_1$.

Theorem 4.1.3. *System of PDEs (4.1.1) with $m \neq 0$ has 34 linearly-independent symmetry operators belonging to the class $\mathcal{M}_1$. The list of these operators is exhausted by the generators of the generalized Galilei group (4.1.3) and by the following 21 operators*:

$$\begin{aligned}
M_1 &= I, \quad M_2 = iI, \\
W_0 &= (1/2)(\gamma_0+\gamma_4)\partial_t - (im/2)(\gamma_0-\gamma_4), \\
W_a &= (1/2)\varepsilon_{abc}\Big((1/2)(\gamma_0+\gamma_4)(\gamma_b\partial_c - \gamma_c\partial_b) + im\gamma_b\gamma_c\Big), \\
S_a &= \gamma_0\gamma_4\partial_a + (\gamma_0+\gamma_4)\gamma_a\partial_t - im(\gamma_0-\gamma_4)\gamma_a, \\
T_a &= (1/2)\varepsilon_{abc}\Big((1/2)(\gamma_0-\gamma_4)(\gamma_b\partial_c - \gamma_c\partial_b) + \gamma_b\gamma_c\partial_t\Big), \\
R_0 &= tW_0 + x_a W_a + (3/4)(\gamma_0+\gamma_4), \\
R_a &= 2tT_a + 2x_a W_0 + \varepsilon_{abc}\Big(x_b S_c + (1/2)\gamma_b\gamma_c\Big) + (3/2)\gamma_a, \\
N_0 &= x_a S_a + \gamma_0\gamma_4,
\end{aligned}$$

$$\begin{aligned}
N_a &= tS_a + 2\varepsilon_{abc}x_bW_c + (\gamma_0+\gamma_4)\gamma_a,\\
K_a &= 2x_aR_0 - (x_bx_b)W_a + \varepsilon_{abc}\Big(tx_bS_c + (1/2)t\gamma_b\gamma_c + (\gamma_0+\gamma_4)x_b\gamma_c\Big)\\
&\quad + t^2T_a + (3/2)t\gamma_a,
\end{aligned}$$

where I is the unit (4×4)-matrix.

Theorem 4.1.4. *Basis of the infinite-dimensional vector space of symmetry operators of system (4.1.1) with* $m=0$ *belonging to the class* $\mathcal{M}_1$ *can be chosen as follows*

$$\begin{aligned}
I_1 &= I,\quad I_2 = iI,\quad A_\infty,\quad G_\infty,\quad D_\infty,\quad T_\infty,\quad J_\infty,\\
W_\infty &= (\gamma_0+\gamma_4)(\varphi_0^6\partial_t + \varphi_a^6\partial_a),\\
S_\infty &= \varphi_a^7\Big((\gamma_0+\gamma_4)\gamma_a\partial_t + \gamma_0\gamma_4\partial_a\Big) + (1/2)(\gamma_0+\gamma_4)\gamma_a\dot{\varphi}_a^7,\\
P_\infty &= \varphi_a^8\Big(2\gamma_a\partial_t - (\gamma_0-\gamma_4)\partial_a\Big) + (1/4)(2\gamma_a + \varepsilon_{abc}\gamma_b\gamma_c)\dot{\varphi}_a^8,\\
Q_\infty &= \varphi_0^9(\gamma_0+\gamma_4)x_a\partial_a,\\
R_\infty &= \varphi_a^{10}\Big\{\varepsilon_{abc}\Big((\gamma_0+\gamma_4)x_b\gamma_c + \gamma_0\gamma_4x_b\partial_c\Big) + (1/2)\gamma_a\Big\}\\
&\quad -(1/2)(\gamma_0+\gamma_4)\gamma_a\dot{\varphi}_a^{10}\gamma_bx_b,\\
N_\infty &= \varphi_0^{11}\Big((\gamma_0+\gamma_4)\gamma_ax_a\partial_t + \gamma_0\gamma_4x_a\partial_a\\
&\quad +(1/2)(1+\gamma_0\gamma_4)\Big) + (1/2)\dot{\varphi}_0^{11}(\gamma_0+\gamma_4)\gamma_ax_a,\\
K_\infty &= \varphi_a^{12}\Big(-(\gamma_0+\gamma_4)(x_bx_b)\partial_a + 2x_a(\gamma_0+\gamma_4)x_b\partial_b\\
&\quad +2x_a(\gamma_0+\gamma_4) + \varepsilon_{abc}x_b\gamma_c\Big),\\
L_\infty &= \varepsilon_{abc}\varphi_a^{13}(\gamma_0+\gamma_4)\Big(x_b\partial_c + (1/4)\gamma_b\gamma_c\Big).
\end{aligned}$$

Here $A_\infty,\ldots,J_\infty$ are operators listed in (4.1.7), φ_μ^N, $\mu=0,\ldots,3$, $N=6,\ldots,13$ are arbitrary smooth functions of t, an overdot means differentiation with respect to t.

Proof. We give the main idea of the proof omitting very cumbersome intermediate calculations. According to the definition of a symmetry operator, to describe all linearly independent symmetry operators of equation (4.1.1) belonging to the class $\mathcal{M}_1$ it is necessary to construct a general solution of the operator equation

$$[L,\,X] = (R_0\partial_t + R_a\partial_a + R)L,$$

where $L = -i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)$; $R_0,\ R_a,\ R$ are variable (4×4)-matrices.

Computing the commutator and equating coefficients of linearly-independent operators ∂_t^2, $\partial_t\partial_a$, $\partial_a\partial_b$, ∂_t, ∂_a yield a system of matrix PDEs for A_0, A_b, B, R_0, R_a, R. Eliminating matrix functions R_0, R_a, R we arrive at the over-determined system of PDEs for 80 functions $A_0^{\mu\nu}$, $A_b^{\mu\nu}$, $B^{\mu\nu}$ (by $A^{\mu\nu}$ we designate the entries of the matrix A) $\mu, \nu = 0, \ldots, 3$ which general solution gives rise to a complete set of symmetry operators of equation (4.1.1). $\rhd$

The complete set of symmetry operators of equation (4.1.1) belonging to the class $\mathcal{M}_1$ does not form a Lie algebra. But it contains some subsets which have very interesting algebraic properties. In particular, the basis generators of the Galilei group P_0, P_a, J_{ab}, G_a, M are even basis elements and the operators W_0, W_a are odd basis elements of a superalgebra. This superalgebra can be considered as a superextension of the Galilei algebra $AG(1,3)$ [305].

A detailed account of symmetry properties of system of linear PDEs (4.1.1) in the class of differential operators of the order higher than 1 and in the class of integro-differential operators can be found in [119].

3. Nonlinear spinor equations invariant under the group $G(1,3)$. In this subsection we will obtain a complete description of Galilei-invariant systems of PDEs

$$\{-i(\gamma_0 + \gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0 - \gamma_4)\}\psi = F(\psi^*, \psi), \tag{4.1.15}$$

where $F(\psi^*, \psi)$ is a complex-valued four-component function. In addition, all the functions $F(\psi^*, \psi)$ such that equation (4.1.15) admits wider symmetry groups (in particular, the generalized Galilei group $G_2(1,3)$) will be constructed.

Theorem 4.1.5. *The system of nonlinear PDEs (4.1.15) is invariant under the Galilei group iff*

$$F(\psi^*, \psi) = \Big(f_1 + (\gamma_0 + \gamma_4)f_2\Big)\psi, \tag{4.1.16}$$

where f_1, f_2 *are arbitrary smooth functions of* $w_1 = \bar{\psi}\psi$, $w_2 = \psi^\dagger\psi + \bar{\psi}\gamma_4\psi$.

Proof. It is convenient to represent a four-component function $F(\psi^*, \psi)$ in the form $F = H(\psi^*, \psi)\psi$, where H is a variable (4×4)-matrix.

At first, we select from the class of equations (4.1.15) those which are invariant under the rotation group $O(3) \subset G(1,3)$. Acting by the first prolongation of the generator of the group $O(3)$

$$Q = \alpha_{ab}x_a\partial_b - (1/4)\{\alpha_{ab}\gamma_a\gamma_b\psi\}^\alpha\partial_{\psi^\alpha} - (1/4)\{\alpha_{ab}\gamma_a^*\gamma_b^*\psi^*\}^\alpha\partial_{\psi^{*\alpha}},$$

where $\alpha_{ab} = -\alpha_{ba}$ are real constants, on (4.1.15) we arrive at the following relation for $H = H(\psi^*, \psi)$:

$$\widetilde{Q}\{-i(\gamma_0 + \gamma_4)\psi_t + i\gamma_a\psi_{x_a} + m(\gamma_0 - \gamma_4)\psi - H\psi\}\Big|_{[L]} = 0. \tag{4.1.17}$$

Here $[L]$ is the set of solutions of PDE (4.1.15).

Designating

$$Q_{ab} = -(1/2)\{\gamma_a\gamma_b\psi\}^\alpha\partial_{\psi^\alpha} - (1/2)\{\gamma_a^*\gamma_b^*\psi^*\}^\alpha\partial_{\psi^{\alpha*}},$$

we rewrite equation (4.1.17) as follows

$$Q_{ab}H + (1/2)[\gamma_a\gamma_b,\, H] = 0. \tag{4.1.18}$$

Expanding the matrix H in the complete system of the Dirac matrices

$$\begin{aligned} H \;=\; & \tilde{a}(\psi^*, \psi)I + b_\mu(\psi^*, \psi)\gamma^\mu + c_{\mu\nu}(\psi^*, \psi)\gamma^\mu\gamma^\nu \\ & + d_\mu(\psi^*, \psi)\gamma_4\gamma^\mu + e(\psi^*, \psi)\gamma_4 \end{aligned} \tag{4.1.19}$$

and substituting it into (4.1.18) we get

$$\begin{aligned} & Q_{ab}(\tilde{a}I + b_\mu\gamma^\mu + c_{\mu\nu}\gamma^\mu\gamma^\nu + d_\mu\gamma_4\gamma^\mu + e\gamma_4) \\ & \quad = b^\mu(g_{\mu a}\gamma_b - g_{\mu b}\gamma_a) + d^\mu\gamma_4(g_{\mu a}\gamma_b - g_{\mu b}\gamma_a) \\ & \quad - c^{\mu\nu}(g_{a\nu}\gamma_b\gamma_\nu + g_{b\mu}\gamma_a\gamma_\nu - g_{a\mu}\gamma_b\gamma_\nu - g_{b\nu}\gamma_a\gamma_\mu). \end{aligned}$$

Equating the coefficients of linearly independent matrices we arrive at the following system of PDEs:

$$\begin{aligned} & Q_{ab}\tilde{a} = Q_{ab}e = Q_{ab}b_0 = Q_{ab}d_0 = 0, \\ & Q_{ab}b_k = b_c(g_{ca}\delta_{kb} - g_{cb}\delta_{ka}), \\ & Q_{ab}d_k = d_c(g_{ca}\delta_{kb} - g_{cb}\delta_{ka}), \\ & Q_{ab}c_{0k} = c_{0c}(g_{ca}\delta_{kb} - g_{cb}\delta_{ka}), \\ & Q_{ab}c_{kc} = c_{mn}(g_{an}\delta^{kc}_{bm} + g_{bm}\delta^{kc}_{an} - g_{am}\delta^{kc}_{bn} - g_{bn}\delta^{kc}_{am}). \end{aligned} \tag{4.1.20}$$

In (4.1.20) $\delta^{kc}_{mn} = \delta_{mk}\delta_{nc} - \delta_{nk}\delta_{mc}$, $a, b, c,\ k, m, n = 1, 2, 3$.

Integration of system (4.1.20) is carried out in the same way as integration of (1.2.6)–(1.2.9). That is why we omit intermediate computations and give

the final result

$$\begin{array}{rcl}\tilde{a} &=& E_1(\vec{y}), \quad b_0 \;=\; E_2(\vec{y}), \quad d_0 \;=\; E_3(\vec{y}), \quad e \;=\; E_4(\vec{y}),\\ b_k &=& \psi^\dagger\gamma_k\psi B_1(\vec{y}) + \psi^\dagger\gamma_4\gamma_k\psi B_2(\vec{y}) + \psi^T\gamma_0\gamma_2\gamma_k\psi B_3(\vec{y}),\\ c_{0k} &=& \psi^\dagger\gamma_k\psi C_1(\vec{y}) + \psi^\dagger\gamma_4\gamma_k\psi C_2(\vec{y}) + \psi^T\gamma_0\gamma_2\gamma_k\psi C_3(\vec{y}),\\ d_k &=& \psi^\dagger\gamma_k\psi D_1(\vec{y}) + \psi^\dagger\gamma_4\gamma_k\psi D_2(\vec{y}) + \psi^T\gamma_0\gamma_2\gamma_k\psi D_3(\vec{y}),\\ c_{ab} &=& \psi^\dagger\gamma_a\gamma_b\psi C_4(\vec{y}) + \psi^\dagger\gamma_4\gamma_a\gamma_b\psi C_4(\vec{y}) + \psi^T\gamma_0\gamma_2\gamma_a\gamma_b\psi C_6(\vec{y}),\end{array} \tag{4.1.21}$$

where $B_1,\ B_2,\dots,E_4$ are arbitrary smooth complex-valued functions; $\vec{y}$ is a complete set of functionally-independent invariants of the group $O(3)$ which can be chosen in the form $\vec{y} = (\bar{\psi}\psi,\ \psi^\dagger\psi,\ \psi^\dagger\gamma_4\psi,\ \bar{\psi}\gamma_4\psi,\ \psi^T\gamma_2\psi)$.

Substitution of (4.1.19), (4.1.21) into (4.1.15) gives rise to the following class of $O(3)$-invariant spinor equations:

$$\begin{aligned}&\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)\}\psi\\ &\quad= \Big\{E_1 + \gamma_0 E_2 + \gamma_0\gamma_4 E_3 + \gamma_4 E_4 + \gamma_a(\psi^\dagger\gamma_a\psi B_1\\ &\quad+\psi^\dagger\gamma_4\gamma_a\psi B_2 + \psi^T\gamma_0\gamma_2\gamma_a\psi B_3) + \gamma_4\gamma_a(\psi^\dagger\gamma_a\psi D_1\\ &\quad+\psi^\dagger\gamma_4\gamma_a\psi D_2 + \psi^T\gamma_0\gamma_2\gamma_a\psi D_3) + \gamma_0\gamma_a(\psi^\dagger\gamma_a\psi C_1\\ &\quad+\psi^\dagger\gamma_4\gamma_a\psi C_2 + +\psi^T\gamma_0\gamma_2\gamma_a\psi C_3) + \gamma_a\gamma_b(\psi^\dagger\gamma_a\gamma_b\psi C_4\\ &\quad+\psi^\dagger\gamma_4\gamma_a\gamma_b\psi C_5 + \psi^T\gamma_0\gamma_2\gamma_a\gamma_b\psi C_6\Big\}\psi.\end{aligned} \tag{4.1.22}$$

Formulae (4.1.22) are substantially simplified if we use identity (1.2.18) rewritten in the form

$$(\bar{\psi}_1\gamma_a\psi_2)\gamma_a\psi_2 = (\psi_1^\dagger\psi_2)\gamma_0\psi_2 - (\bar{\psi}_1\psi_2)\psi_2 - (\bar{\psi}_1\gamma_4\psi_2)\gamma_4\psi_2.$$

Here $\psi_1,\ \psi_2$ are arbitrary four-component functions.

Due to the above identity equation (4.1.22) takes a more compact form

$$\begin{aligned}&\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)\}\psi\\ &\quad= (h_1 + h_2\gamma_0 + h_3\gamma_0\gamma_4 + h_4\gamma_4)\psi,\end{aligned} \tag{4.1.23}$$

where $h_i = h_i(y_1,\ y_2,\ y_3,\ y_4,\ y_5)$, $i = 1,\dots,4$ are arbitrary smooth complex-valued functions.

Next, acting with the first prolongation of the generator of group (4.1.11) on (4.1.23) and using the Lie invariance criterion we get the following equations for $H = h_1 + h_2\gamma_0 + h_3\gamma_0\gamma_4 + h_4\gamma_4$:

$$Q_a H + (1/2)[H,\ (\gamma_0+\gamma_4)\gamma_a] = 0, \quad a = 1,2,3, \tag{4.1.24}$$

where $Q_a = -(1/2)\{(\gamma_0+\gamma_4)\gamma_a\psi\}^\alpha\partial_{\psi^\alpha} - (1/2)\{(\gamma_0^*+\gamma_4^*)\gamma_a^*\psi^*\}^\alpha\partial_{\psi^{*\alpha}}$.

Computing commutators in the left-hand sides of system (4.1.24) and equating to zero the coefficients of linearly independent γ-matrices we come to the system of PDEs for $h_1,\ h_2,\ h_3,\ h_4$

$$Q_a h_1 = 0, \quad Q_a h_2 = 0, \quad h_2 = h_4, \quad h_3 = 0. \tag{4.1.25}$$

Integration of the above equations yields

$$h_1 = f_1(w_1,\, w_2), \quad h_2 = h_4 = f_2(w_1,\, w_2), \quad h_3 = 0,$$

where $w_1 = \bar\psi\psi,\ \ w_2 = \psi^\dagger\psi + \bar\psi\gamma_4\psi$.

Generally speaking, the group $G(1,3)$ is not the maximal invariance group of equation (4.1.15) with F of the form (4.1.16). Below we give without proof the assertions describing functions $F = F(\psi^*,\, \psi)$ such that the system of PDEs (4.1.15) admits wider groups.

Theorem 4.1.6. *Equation (4.1.15) is invariant under the group* $G_1(1,3) = G(1,3)\otimes D(1)$*, where* $D(1)$ *is the one-parameter group of scale transformations, only in the following cases:*

1) $f_1 = (\psi^\dagger\psi + \bar\psi\gamma_4\psi)^{1/(2k-1)}\tilde f_1\Big((\bar\psi\psi)^{1-2k}(\psi^\dagger\psi+\bar\psi\gamma_4\psi)^{2k}\Big),$

$$f_2 = (\psi^\dagger\psi + \bar\psi\gamma_4\psi)^{2/(2k-1)}\tilde f_2\Big((\bar\psi\psi)^{1-2k}(\psi^\dagger\psi+\bar\psi\gamma_4\psi)^{2k}\Big), \tag{4.1.26}$$

$D(1)$ being of the form

$t' = te^{2\theta_0}, \quad x'_a = x_a e^{\theta_0},$

$$\psi'(t',\, \vec x') = \exp\Big\{\theta_0\Big(-k + (1/2)\gamma_0\gamma_4\Big)\Big\}\psi(t,\, \vec x), \tag{4.1.27}$$

under $k \neq 1/2$;

2) $f_1 = \bar\psi\psi\tilde f_1(\psi^\dagger\psi + \bar\psi\gamma_4\psi), \quad f_2 = (\bar\psi\psi)^2\tilde f_2(\psi^\dagger\psi + \bar\psi\gamma_4\psi),$

$D(1)$ being of the form (4.1.27) *under* $k = 1/2$;

3) $m = 0, \quad f_1 = (\psi^\dagger\psi + \bar\psi\gamma_4\psi)^{1/2k}\tilde f_1\Big((\bar\psi\psi)(\psi^\dagger\psi+\bar\psi\gamma_4\psi)^{-1}\Big),$

$$f_2 = (\psi^\dagger\psi + \bar\psi\gamma_4\psi)^{1/2k}\tilde f_2\Big((\bar\psi\psi)(\psi^\dagger\psi+\bar\psi\gamma_4\psi)^{-1}\Big), \tag{4.1.28}$$

$D(1)$ being of the form

$$t' = te^{\theta_0}, \quad x'_a = x_a e^{\theta_0}, \quad \psi'(t',\, \vec x') = e^{-k\theta_0}\psi(t,\, \vec x). \tag{4.1.29}$$

Theorem 4.1.7. *Equation (4.1.15) is invariant under the generalized Galilei group* $G_2(1,3)$ *only in the following cases:*

1) $f_1,\ f_2$ are of the form (4.1.26) under $k = 3/2$, the groups of scale and projective transformations are given by formulae (4.1.27) (with $k = 3/2$) and (4.1.13);

2) $m = 0,\ f_1,\ f_2$ are of the form (4.1.28) under $k = 3/2$, the group of scale transformations is of the form (4.1.29) with $k = 3/2$ and the group of the projective transformations has the form

$$t' = t(1-\theta_0 t)^{-1}, \quad x'_a = x_a(1-\theta_0 t)^{-2},$$
$$\psi'(t',\vec{x}') = (1-\theta_0 t)^3 \exp\{\theta_0(1-\theta_0 t)^{-1}(\gamma_0+\gamma_4)\gamma_a x_a\}\psi(t,\vec{x}).$$

4.2. Exact solutions of Galilei-invariant spinor equations

The present section is devoted to reduction and construction of the multi-parameter families of exact solutions of the nonlinear Galilei-invariant systems of PDEs

$$\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4) - f_1 - f_2(\gamma_0+\gamma_4)\}\psi = 0, \qquad (4.2.1)$$

where $f_i = f_i(\bar{\psi}\psi,\ \psi^{\dagger}\psi + \bar{\psi}\psi)$.

1. Ansätze for the spinor field. Since a linear representation of the Galilei algebra is realized on the set of solutions of the system of PDEs (4.2.1), we can look for Ansätze reducing (4.2.1) to systems of ODEs in the form

$$\psi(t,\vec{x}) = A(t,\vec{x})\varphi(\omega), \qquad (4.2.2)$$

where $\varphi = \varphi(\omega)$ is a complex-valued four-component function. A variable (4×4)-matrix $A(t,\vec{x})$ and a real-valued function $\omega = \omega(t,\vec{x})$ are determined by equations (1.5.22), (1.5.20), where operators $Q_1,\ Q_2,\ Q_3$ are the basis elements of some three-dimensional subalgebra of the Galilei algebra $AG(1,3)$.

A classification of the $G(1,3)$ non-conjugate subalgebras of the algebra $AG(1,3)$ has been carried out in [267] (we use a more convenient classification given in [100]). Each three-dimensional subalgebra $\langle Q_1, Q_2, Q_3\rangle$ satisfying condition (1.5.10) gives rise to an Ansatz of the form (4.2.2) which reduces the $G(1,3)$-invariant system of PDEs (4.2.1) to a system of ODEs for $\varphi(\omega)$ (Theorem 1.5.1).

It should be noted that the subalgebraic structure of the algebra $AG(1,3)$ in the case $m \neq 0$ differs essentially from the one in the case $m = 0$. That is why the cases $m \neq 0$ and $m = 0$ lead to principally different sets of Ansätze for the spinor field.

Since the system of nonlinear PDEs (4.2.1) with $m = 0$ admits the infinite-parameter Lie group with generators $G_\infty,\ J_\infty$ from (4.1.7) [292], which contains the group $G(1,3)$ as a subgroup, it makes no sense reducing it by means of subgroups of the Galilei groups. That is why we restrict ourselves to the case $m \neq 0$ (Galilei-invariant Ansätze for the case $m = 0$ are constructed in [160]).

At first, we will write down the complete list of inequivalent Ansätze for the spinor field invariant under the $G(1,3)$ non-conjugate three-dimensional subalgebras of the algebra $AG(1,3)$ and then consider an example of integration of the over-determined system of equations (1.5.22), (1.5.20).

$$
\begin{aligned}
&1) \quad \langle P_0,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \varphi\,(x_3);\\
&2) \quad \langle J_{12} + \alpha P_0,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{(t/2\alpha)\gamma_1\gamma_2\}\varphi\,(x_3);\\
&3) \quad \langle P_0 + i\alpha m,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{-i\alpha m t\}\varphi\,(x_3);\\
&4) \quad \langle J_{12},\, P_0,\, P_3\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{-(1/2)\gamma_1\gamma_2 \arctan(x_1/x_2)\}\varphi\,(x_1^2 + x_2^2);\\
&5) \quad \langle J_{12} + \alpha P_0 + \beta G_3,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{(2im/3)\beta\alpha^{-2}t(\beta t^2 - 3\alpha x_3) - (\beta t/\alpha)\eta_3 + (t/2\alpha)\gamma_1\gamma_2\}\\
&\quad\ \ \times\varphi\,(\beta t^2 - 2\alpha x_3);\\
&6) \quad \langle J_{12} + \alpha G_3,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{(1/2\alpha t)x_3(\gamma_1\gamma_2 - 2\alpha\eta_3 - 2im\alpha x_3)\}\varphi\,(t);\\
&7) \quad \langle J_{12} + \alpha G_3,\, G_1,\, G_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{-(im/t)(x_1^2 + x_2^2) - (1/t)(\eta_1 x_1 + \eta_2 x_2)\}\\
&\quad\ \ \times\exp\{(1/2\alpha t)x_3(2i\alpha m x_3 + \alpha\eta_3 - \gamma_1\gamma_2)\}\varphi\,(t);\\
&8) \quad \langle J_{12} + \alpha P_3,\, P_1,\, P_2\rangle,\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{-(1/2\alpha)x_3\gamma_1\gamma_2\}\varphi\,(t);\\
&9) \quad \langle J_{12} + \alpha P_3,\, G_1,\, G_2\rangle, \qquad\qquad\qquad\qquad\qquad\qquad (4.2.3)\\
&\quad\ \ \psi(t,\,\vec{x}) = \exp\{-(im/t)(x_1^2 + x_2^2) - (1/t)(\eta_1 x_1 + \eta_2 x_2)\}
\end{aligned}
$$

$\times \exp\{-(1/2\alpha)x_3\gamma_1\gamma_2\}\varphi(t);$

10) $\langle P_1,\, P_2,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \varphi(t);$

11) $\langle G_1,\, P_2,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{-(im/t)x_1^2 - (1/t)x_1\eta_1\}\varphi(t);$

12) $\langle G_1 + \alpha P_1,\, G_2,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{-im[t^{-1}x_2^2 + (t-\alpha)^{-1}x_1^2] - t^{-1}x_2\eta_2$
$+(\alpha - t)^{-1}x_1\eta_1\}\varphi(t);$

13) $\langle G_1 + \alpha P_1,\, G_2 + \beta P_2,\, G_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{im[(\alpha - t)^{-1}x_1^2 + (\beta - t)^{-1}x_2^2 - t^{-1}x_3^2]$
$+(\alpha - t)^{-1}x_1\eta_1 + (\beta - t)^{-1}x_2\eta_2 - t^{-1}x_3\eta_3\}\varphi(t);$

14) $\langle G_1 + \alpha P_0,\, P_2,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{(2im/3)\alpha^{-2}t(t^2 - 3\alpha x_1) - (t/\alpha)\eta_1\}\varphi(t);$

15) $\langle J_{12} + i\alpha m,\, P_0,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{[i\alpha m - (1/2)\gamma_1\gamma_2]\arctan(x_1/x_2)\}\varphi(x_1^2 + x_2^2);$

16) $\langle J_{12} + i\alpha m,\, P_0 + i\beta m,\, P_3\rangle$,
$\psi(t,\,\vec{x}) = \exp\{i\beta mt + [i\alpha m - (1/2)\gamma_1\gamma_2]\arctan(x_1/x_2)\}\varphi(x_1^2 + x_2^2);$

17) $\langle G_1 + \alpha P_2,\, G_2 + \alpha P_1 + \beta P_2 + \tau P_3,\, G_3 - \rho G_1 - \delta G_2 - \alpha\delta P_1\rangle$,
$\psi(t,\,\vec{x}) = \exp\{-(im/t)x_1^2 - (1/t)x_1\eta_1\}\exp\Big\{-(im/t)(\alpha x_1 + tx_2)^2$
$\times[t(t-\beta) - \alpha^2]^{-1} + (1/\tau t)(\alpha\eta_1 + t\eta_2)x_3\Big\}\exp\Big\{imw^2\Big(f(t)[t(t-\beta)$
$-\alpha^2]\Big)^{-1} - [f(t)]^{-1}w\Big((\delta t^{-1} - \tau^{-1})(\alpha\eta_1 + t\eta_2) - \eta_3\Big)\Big\}\varphi(t).$

In the above formulae $\alpha,\ \beta,\ \rho,\ \delta$ are arbitrary real parameters; $\eta_a = (1/2)(\gamma_0 + \gamma_4)\gamma_a$, $a = 1, 2, 3$; $\varphi(\omega)$ is an arbitrary four-component function;

$$w = \tau(\alpha x_1 + tx_2) + \Big(t(t-\beta) - \alpha^2\Big)x_3, \quad \tau = \alpha\rho + \beta\delta,$$
$$f(t) = \tau\Big(\alpha(\rho t - \alpha\delta) + \delta t^2\Big) - t\Big(t(t-\beta) - \alpha^2\Big).$$

As an example, we construct the Ansatz N 11 from (4.2.3). Substitution of $Q_1 = t\partial_{x_1} + 2imx_1 + (1/2)(\gamma_0 + \gamma_4)\gamma_1$, $Q_2 = \partial_{x_2}$, $Q_3 = \partial_{x_3}$ into the system of PDEs (1.5.22), (1.5.20) gives the following equations for $A(t,\,\vec{x})$, $\omega(t,\,\vec{x})$:

$$\begin{aligned} &\partial_{x_2}A = \partial_{x_3}A = 0,\\ &t\partial_{x_1}A + \Big(2imx_1 + (1/2)(\gamma_0 + \gamma_4)\gamma_1\Big)A = 0, \end{aligned} \tag{4.2.4}$$

$$\partial_{x_2}\omega = 0, \quad \partial_{x_3}\omega = 0, \quad t\partial_{x_1}\omega = 0. \tag{4.2.5}$$

The first integral of system (4.2.5) has the form $\omega = t$. Next, from (4.2.4) it follows that $A = A(t,\, x_1)$ and in addition

$$\frac{\partial A}{\partial x_1} = -(1/t)\Big(2imx_1 + (1/2)(\gamma_0 + \gamma_4)\gamma_1\Big)A.$$

Integrating the above equation we get the expression for A.

2. Reduction of nonlinear equation (4.2.1). We will carry out reduction of PDE (4.2.1) to systems of ODEs provided $m \neq 0$. Substitution of Ansätze (4.2.2) into (4.2.1) gives rise to equations of the form

$$A(t,\, \vec{x})L\left(\omega, \varphi^*, \varphi, \frac{d\varphi}{d\omega}\right) = 0. \tag{4.2.6}$$

Since $\det A(t,\, \vec{x}) \neq 0$, the above equation is rewritten as follows

$$L\left(\omega,\, \varphi^*,\, \varphi,\, \frac{d\varphi}{d\omega}\right) = 0.$$

Below we give explicit forms of systems of PDEs for $\varphi = \varphi(\omega)$ corresponding to the Galilei-invariant Ansätze for the spinor field $\psi(t,\, \vec{x})$ listed in (4.2.3)

1) $i\gamma_3\dot{\varphi} + m(\gamma_0 - \gamma_4)\varphi = \widetilde{F}$,

2) $i\gamma_3\dot{\varphi} + \Big((i/2\alpha)(\gamma_0 + \gamma_4)\gamma_3 + m(\gamma_0 - \gamma_4)\Big)\varphi = \widetilde{F}$,

3) $i\gamma_3\dot{\varphi} + \Big(\alpha m(\gamma_0 + \gamma_4) + m(\gamma_0 - \gamma_4)\Big)\varphi = \widetilde{F}$,

4) $2i\omega^{1/2}\gamma_2\dot{\varphi} + \Big((i/2)\omega^{-1/2}\gamma_2 + m(\gamma_0 - \gamma_4)\Big)\varphi = \widetilde{F}$,

5) $-2i\alpha\gamma_3\dot{\varphi} + \Big(m(\gamma_0 - \gamma_4) + \alpha^{-2}\beta\omega(\gamma_0 + \gamma_4)$
$+(i/2\alpha)(\gamma_0 + \gamma_4)\gamma_3\Big)\varphi = \widetilde{F}$,

6) $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \Big(m(\gamma_0 - \gamma_4) + (i/2\alpha\omega)[\gamma_0\gamma_4$
$-\alpha(\gamma_0 + \gamma_4)]\Big)\varphi = \widetilde{F}$,

7) $-i(\gamma_0 + \gamma_4)\dot{\varphi} + \Big((i/2\alpha\omega)\gamma_0\gamma_4 - (i/\omega)(\gamma_0 + \gamma_4) + m(\gamma_0 - \gamma_4)\Big)\varphi$
$= \widetilde{F}$,

$$\begin{aligned}
&8) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(m(\gamma_0-\gamma_4) - (i/2\alpha)\gamma_0\gamma_4\Big)\varphi = \widetilde F,\\
&9) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(m(\gamma_0-\gamma_4) - i\omega^{-1}(\gamma_0+\gamma_4) - i(2\alpha)^{-1}\gamma_0\gamma_4\Big)\varphi\\
&\qquad = \widetilde F,\\
&10) \quad -i(\gamma_0+\gamma_4)\dot\varphi + m(\gamma_0-\gamma_4)\varphi = \widetilde F, \qquad\qquad (4.2.7)\\
&11) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(m(\gamma_0-\gamma_4) - (i/2\omega)(\gamma_0+\gamma_4)\Big)\varphi = \widetilde F,\\
&12) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(m(\gamma_0-\gamma_4) - i(2\omega-\alpha)[2\omega(\omega-\alpha)]^{-1}\\
&\qquad \times(\gamma_0+\gamma_4)\Big)\varphi = \widetilde F,\\
&13) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(m(\gamma_0-\gamma_4) - i[3\omega^2 - 2(\alpha+\beta)\omega + \alpha\beta]\\
&\qquad \times[2\omega(\omega-\alpha)(\omega-\beta)]^{-1}(\gamma_0+\gamma_4)\Big\}\varphi = \widetilde F,\\
&14) \quad -2i\alpha\gamma_1\dot\varphi + \Big(m(\gamma_0-\gamma_4) + m\alpha^{-2}\omega(\gamma_0+\gamma_4)\Big)\varphi = \widetilde F,\\
&15) \quad 2i\omega^{1/2}\gamma_2\dot\varphi + \Big(i\omega^{-1/2}[i\alpha m\gamma_1 + (1/2)\gamma_2] + m(\gamma_0-\gamma_4)\Big)\varphi = \widetilde F,\\
&16) \quad 2i\omega^{1/2}\gamma_2\dot\varphi + \Big(i\omega^{-1/2}[i\alpha m\gamma_1 + (1/2)\gamma_2] + m(\gamma_0-\gamma_4)\\
&\qquad + m\beta(\gamma_0+\gamma_4)\Big)\varphi = \widetilde F,\\
&17) \quad -i(\gamma_0+\gamma_4)\dot\varphi + \Big(i[2\omega f(\omega)]^{-1}[\omega^3 + \alpha(\alpha+\rho\tau)\omega - 2\tau\alpha^2\delta]\\
&\qquad \times(\gamma_0+\gamma_4) - (i/\omega)(\gamma_0+\gamma_4) + m(\gamma_0-\gamma_4)\Big)\varphi = \widetilde F.
\end{aligned}$$

Here a dot over φ means differentiation with respect to ω,

$$\widetilde F = \{f_1(\bar\varphi\varphi,\, \varphi^\dagger\varphi + \bar\varphi\gamma_4\varphi) + (\gamma_0+\gamma_4) f_2(\bar\varphi\varphi,\, \varphi^\dagger\varphi + \bar\varphi\gamma_4\varphi)\}\varphi,$$
$$\tau = \alpha\rho + \delta\beta, \quad f(\omega) = \tau[\alpha(\rho\omega - \alpha\delta) + \delta\omega^2] - [\omega(\omega-\beta) - \alpha^2]\omega.$$

3. Exact solutions of nonlinear equation (4.2.1). We will construct the multi-parameter families of exact solutions of nonlinear Galilei-invariant system of PDEs of the form (4.2.1) with the power nonlinearity

$$\Big\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4) - \lambda(\psi^\dagger\psi + \bar\psi\gamma_4\psi)^{1/2k}\Big\}\psi = 0, \quad (4.2.8)$$

where $\lambda,\ k$ are constants, by means of the Ansätze for the spinor field $\psi(t,\,\vec x)$ invariant under the $G(1,3)$ non-conjugate three-dimensional subalgebras of the Galilei algebra $AG(1,3)$.

According to the results obtained in the previous subsection substitution of Ansätze (4.2.3) into (4.2.8) gives rise to systems of ODEs (4.2.7) with $\widetilde F = \lambda(\varphi^\dagger\varphi + \bar\varphi\gamma_4\varphi)^{1/2k}\varphi$.

ODEs 6–13 prove to be integrable due to the following assertion.

Lemma 4.2.1. *The quantities*

$$\begin{aligned}
&I_6 = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,\omega, \quad I_7 = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,\omega^2,\\
&I_8 = \bar{\varphi}(\gamma_0+\gamma_4)\varphi, \quad I_9 = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,\omega^2,\\
&I_{10} = \bar{\varphi}(\gamma_0+\gamma_4)\varphi, \quad I_{11} = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,\omega,\\
&I_{12} = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,(\omega^2-\alpha\omega),\\
&I_{13} = \bar{\varphi}(\gamma_0+\gamma_4)\varphi\,\omega(\omega-\alpha)(\omega-\beta)
\end{aligned}$$

are the first integrals of the systems of PDEs 6,7,...,13, respectively.

Proof is carried out by a direct check. ▷

According to the above lemma, the systems of nonlinear ODEs 6–13 are linearized. And what is more, the linear systems obtained are integrable in quadratures. This fact enables us to construct the general solutions of the reduced ODEs 6–13 from (4.2.7). These solutions are represented in the following unified form:

$$\varphi_N(\omega) = (1/2)\{f_N(\omega)(\gamma_0+\gamma_4) + g_N(\omega)(1+\gamma_0\gamma_4)\}\chi, \tag{4.2.9}$$

where $N = 6,\ldots,13$, χ is an arbitrary constant four-component column,

$$\begin{aligned}
f_6(\omega) &= (1/2m)[\tilde{\lambda}\,\omega^{-1/2k} - (i/2\alpha\omega)]g_6,\\
g_6(\omega) &= \omega^{-1/2}\exp\{-(i/16)(\alpha^2 m\omega)^{-1} + iW_1(k,\omega)\};\\
f_7(\omega) &= (1/2m)[\tilde{\lambda}\,\omega^{-1/k} - (i/2\alpha\omega)]g_7,\\
g_7(\omega) &= \omega^{-1}\exp\{-(i/16)(\alpha^2 m\omega)^{-1} + iW_2(k,\omega)\};\\
f_8(\omega) &= (1/2m)[\tilde{\lambda} + (i/2\alpha)]g_8,\\
g_8(\omega) &= \exp\{i(1+4\alpha^2\tilde{\lambda}^2)(16\alpha^2 m)^{-1}\omega\};\\
f_9(\omega) &= (1/2m)[\tilde{\lambda}\omega^{-1/k} + (i/2\alpha)]g_9,\\
g_9(\omega) &= (1/\omega)\exp\{i\omega(16\alpha^2 m)^{-1} + iW_2(k,\omega)\};\\
f_{10}(\omega) &= (\tilde{\lambda}/2m)\exp\{(i/4m)\tilde{\lambda}^2\omega\},\\
g_{10}(\omega) &= \exp\{(i/4m)\tilde{\lambda}^2\omega\};\\
f_{11}(\omega) &= (\tilde{\lambda}/2m)\omega^{1/2k}g_{11},\\
g_{11}(\omega) &= \omega^{1/2}\exp\{iW_1(k,\omega)\};\\
f_{12}(\omega) &= (\tilde{\lambda}/2m)(\omega^2-\alpha\omega)^{-1/2k}g_{12},\\
g_{12}(\omega) &= (\omega^2-\alpha\omega)^{-1/2}\exp\left\{i\tilde{\lambda}^2(4m)^{-1}\int^{\omega}(z^2-\alpha z)^{-1/k}dz\right\};
\end{aligned} \tag{4.2.10}$$

$$\begin{aligned}
f_{13}(\omega) &= (\tilde{\lambda}/2m)[\omega(\omega-\alpha)(\omega-\beta)]^{-1/2k}g_{13},\\
g_{13}(\omega) &= [\omega(\omega-\alpha)(\omega-\beta)]^{-1/2}\\
&\quad\times\exp\left\{(i\tilde{\lambda}^2/4m)\int^{\omega}[z(z-\alpha)(z-\beta)]^{-1/k}dz\right\}.
\end{aligned}$$

In (4.2.10) $\tilde{\lambda}=\lambda(\chi^{\dagger}\chi+\bar{\chi}\gamma_4\chi)^{1/2k}$,

$$W_n(k,\omega)=(\tilde{\lambda}^2/4m)\begin{cases} k(k-n)^{-1}\omega^{(k-n)/k}, & \text{under } k\neq n,\\ \ln\omega, & \text{under } k=n.\end{cases}$$

Substitution of the above results into the corresponding Ansätze for the spinor field $\psi(t,\vec{x})$ yields the following classes of exact solutions of nonlinear equation (4.2.8):

$$\begin{aligned}
\psi(t,\vec{x}) &= \exp\{x_3(2\alpha t)^{-1}(\gamma_1\gamma_2-2\alpha\eta_3-2i\alpha m x_3)\}\varphi_6(t),\\
\psi(t,\vec{x}) &= \exp\{-imt^{-1}x_ax_a-t^{-1}\eta_ax_a\}\exp\{(2\alpha t)^{-1}x_3\gamma_1\gamma_2\}\varphi_7(t),\\
\psi(t,\vec{x}) &= \exp\{-(2\alpha)^{-1}x_3\gamma_1\gamma_2\}\varphi_8(t),\\
\psi(t,\vec{x}) &= \exp\{-imt^{-1}(x_1^2+x_2^2)-t^{-1}(x_1\eta_1+x_2\eta_2)\}\\
&\quad\times\exp\{-(2\alpha)^{-1}x_3\gamma_1\gamma_2\}\varphi_9(t),\\
\psi(t,\vec{x}) &= \varphi_{10}(t), \qquad\qquad (4.2.11)\\
\psi(t,\vec{x}) &= \exp\{-imt^{-1}x_1^2-t^{-1}x_1\eta_1\}\varphi_{11}(t),\\
\psi(t,\vec{x}) &= \exp\{-imt^{-1}x_2^2-t^{-1}x_2\eta_2\}\exp\{im(\alpha-t)^{-1}x_1^2\\
&\quad+(\alpha-t)^{-1}x_1\eta_1\}\varphi_{12}(t),\\
\psi(t,\vec{x}) &= \exp\{-im[t^{-1}x_3^2+(t-\alpha)^{-1}x_1^2+(t-\beta)^{-1}x_2^2]\\
&\quad-t^{-1}x_3\eta_3+(\alpha-t)^{-1}x_1\eta_1+(\beta-t)^{-1}x_2\eta_2\}\varphi_{13}(t).
\end{aligned}$$

To obtain $G(1,3)$-ungenerable families of exact solutions of system of nonlinear PDEs (4.2.8) it is necessary to apply the procedure of generating solutions by transformations from the symmetry group of system of PDEs (4.2.8).

Using Theorem 2.4.1 it is not difficult to obtain the formulae of generating solutions by transformation groups (4.1.9)–(4.1.14)

$$P\ :\ \begin{cases}\psi_{II}(t,\vec{x})=\psi_I(t',\vec{x}'),\\ t'=t+\theta_0,\\ x'_a=x_a+\theta_a;\end{cases}$$

$$J \quad : \quad \begin{cases} \psi_{II}(t,\,\vec{x}) = \exp\{(1/4)\varepsilon_{abc}\theta_a\gamma_b\gamma_c\}\psi_I(t',\,\vec{x}'), \\ t' = t, \\ x'_a = \Big(\delta_{ab}\cos\theta + \varepsilon_{abc}\theta_c\theta^{-1}\sin\theta \\ \quad +\theta_a\theta_b\theta^{-2}(1-\cos\theta)\Big)x_b; \end{cases}$$

$$G \quad : \quad \begin{cases} \psi_{II}(t,\,\vec{x}) = \exp\Big\{2im\Big(\theta_a x_a + (t/2)\theta_a\theta_a\Big) \\ \quad +(1/2)(\gamma_0+\gamma_4)\gamma_a\theta_a\Big\}\psi_I(t',\,\vec{x}'), \\ t' = t, \\ x'_a = x_a + \theta_a t; \end{cases}$$

$$D \quad : \quad \begin{cases} \psi_{II}(t,\,\vec{x}) = \exp\{2\theta_0 - (1/2)\theta_0\gamma_0\gamma_4\}\psi_I(t',\,\vec{x}'), \\ t' = te^{2\theta_0}, \\ x'_a = x_a e^{\theta_0}; \end{cases}$$

$$A \quad : \quad \begin{cases} \psi_{II}(t,\,\vec{x}) = (1-\theta_0 t)^{-2}\exp\Big\{im\theta_0(1-\theta_0 t)^{-1}x_a x_a \\ \quad +(1/2t)\ln(1-\theta_0 t)\Big(t\gamma_0\gamma_4 + (\gamma_0+\gamma_4)\gamma_a x_a\Big)\Big\}\psi_I(t',\,\vec{x}'), \\ t' = t(1-\theta_0 t)^{-1}, \\ x'_a = x_a(1-\theta_0 t)^{-1}; \end{cases}$$

$$M \quad : \quad \begin{cases} \psi_{II}(t,\,\vec{x}) = e^{2im\theta_0}\psi_I(t',\,\vec{x}'), \\ t' = t, \\ x'_a = x_a, \end{cases}$$

where $\theta_0,\ \theta_a$ are arbitrary parameters, $\theta = (\theta_a\theta_a)^{1/2}$.

Applying the solution generation formulae to (4.2.11) and making some rather cumbersome computations yield the $G(1,3)$-ungenerable families of exact solutions of nonlinear equation (4.2.8). Below we present one of them

$$\begin{aligned} \psi(t,\,\vec{x}) \quad = \quad & (1/2)\exp\{im(2\theta_a x_a + t\theta_a\theta_a)\}\exp\Big\{-(1/2T)a_b z_b\Big(2ima_b z_b \\ & +(\gamma_0+\gamma_4)\gamma_b a_b\Big)\Big\}\Big\{(\gamma_0+\gamma_4)\Big(f_{11}(T) + g_{11}(T)\gamma_a\theta_a\Big) \\ & +g_{11}(T)(1+\gamma_0\gamma_4)\Big\}\chi, \end{aligned}$$

where $z_a = x_a + t\theta_a + \tau_a$, $T = t + \tau_0$; $\{\theta_a,\ \tau_\mu\} \subset \mathbb{R}^1$ are arbitrary parameters; $\vec{a}$ is an arbitrary constant unit vector; the functions $f_{11}(T)$, $g_{11}(T)$ are given in (4.2.10).

It is worth noting that all solutions (4.2.11) have a singularity at the point $m = 0$. Consequently, it is impossible to obtain solutions of massless equation (4.1.1) putting in (4.2.11) $m = 0$.

4.3. Galilei-invariant second-order spinor equations

As noted in Section 4.1 the substitution

$$\psi(t,\,\vec{x}) = \{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)\}\Psi(t,\,\vec{x}) \tag{4.3.1}$$

reduces equation (4.1.1) to a system of splitting Schrödinger equations

$$(4im\partial_t - \partial_a\partial_a)\Psi^\alpha(t,\,\vec{x}) = 0, \tag{4.3.2}$$

where Ψ^α are components of the spinor Ψ.

Thus, the problem of integration of system of linear PDEs (4.1.1) is reduced to the integration of the scalar Schrödinger equation. That is why system of the second-order PDEs (4.3.2) can also be used to describe a Galilean particle with spin $s = 1/2$. However equations (4.3.2) describe particles with different spins because they are invariant under the Galilei group having the generators

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_a, \\ &J_{ab} = -x_a\partial_b + x_b\partial_a + S_{ab}, \quad a \neq b, \\ &G_a = t\partial_a + 2imx_a + \eta_a, \end{aligned} \tag{4.3.3}$$

where $S_{ab},\ \eta_a$ are arbitrary constant matrices of the corresponding dimensions which satisfy the commutation relations

$$\begin{aligned} &[S_{ab},\,S_{cd}] = -\delta_{ad}S_{bc} - \delta_{bc}S_{ad} + \delta_{ac}S_{bd} + \delta_{bd}S_{ac}, \\ &[\eta_a,\,S_{bc}] = \delta_{ac}\eta_b - \delta_{ab}\eta_c, \quad [\eta_a,\,\eta_b] = 0. \end{aligned} \tag{4.3.4}$$

To obtain from (4.3.2) a system of PDEs describing a particle with spin $s = 1/2$ one should impose an additional constraint on the set of solutions of (4.3.2). For example, if equations (4.3.2) are considered together with (4.1.1)

$$\begin{aligned} &(4im\partial_t - \partial_a\partial_a)\psi(t,\,\vec{x}) = 0, \\ &\{-i(\gamma_0+\gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0-\gamma_4)\}\psi(t,\,\vec{x}) = 0, \end{aligned} \tag{4.3.5}$$

then the maximal (in Lie sense) invariance group of the system obtained is the generalized Galilei group having the generators (4.1.3). This assertion follows from the fact that the set of solutions of system (4.3.5) coincides with the set of solutions of equation (4.1.1).

Imposing on solutions of system (4.3.2) the weaker nonlinear constraint

$$\partial_t\Big(\bar{\psi}(\gamma_0+\gamma_4)\psi\Big) = \partial_a(\bar{\psi}\gamma_a\psi) \tag{4.3.6}$$

we get another example of a Galilei-invariant system of PDEs for a particle with spin $s = 1/2$. Let us note that additional constraint (4.3.6) is an algebraic consequence of equation (4.1.1). Therefore, the set of solutions of system (4.3.2), (4.3.6) contains all solutions of equation (4.1.1).

By a direct check we can become convinced of the fact that the system of PDEs (4.3.2) is not invariant under the generalized Galilei group with generators (4.1.3). The same assertion holds for nonlinear equations of the form

$$(4im\partial_t - \partial_a\partial_a)\psi + F(\psi^*, \psi) = 0, \tag{4.3.7}$$

where F is a complex-valued four-component function.

Theorem 4.3.1. *The system of PDEs (4.3.7) is invariant under the Galilei group with the generators $P_0,\ P_a,\ J_{ab},\ G_a,\ M$ from (4.1.3) iff*

$$F = \{f_1 + (\gamma_0 + \gamma_4)f_2\}\psi, \tag{4.3.8}$$

where $f_1,\ f_2$ are arbitrary smooth functions of $w_1 = \bar{\psi}\psi$, $w_2 = \psi^\dagger\psi + \bar{\psi}\gamma_4\psi$. Furthermore, the class of PDEs (4.3.7) contains no equations admitting the group $G_2(1,3)$ with generators (4.1.3).

Proof. Invariance of system (4.3.7) with respect to the group of translations (4.1.9) is evident. Consequently, to prove the theorem we have to study the restrictions imposed on the four-component function $F(\psi^*,\, \psi)$ by the requirement that (4.3.7) admits the Lie groups with the generators $J_{ab},\ G_a$. Acting by the first prolongation of the operators $J_{ab},\ G_a$ on (4.3.7) and applying the Lie invariance criterion we get an over-determined system of linear PDEs for $F(\psi^*,\, \psi)$. If we rewrite the function F in the equivalent form $H(\psi^*,\, \psi)\psi$, where $H(\psi^*,\, \psi)$ is a 4×4-matrix, then the system of PDEs in question takes the form

$$\begin{aligned}
&\Big(\{\gamma_a\gamma_b\psi\}^\alpha\partial_{\psi^\alpha} + \{\gamma_a^*\gamma_b^*\psi^*\}^\alpha\partial_{\psi^{\alpha*}}\Big)H = [\gamma_a\gamma_b,\, H],\\
&\Big(\{(\gamma_0+\gamma_4)\gamma_a\psi\}^\alpha\partial_{\psi^\alpha} + \{(\gamma_0^*+\gamma_4^*)\gamma_a^*\psi^*\}^\alpha\partial_{\psi^{*\alpha}}\Big)H\\
&\quad = [H,\, (\gamma_0+\gamma_4)\gamma_a].
\end{aligned} \tag{4.3.9}$$

Here $\{\psi\}^\alpha$ is the α-th component of ψ, $[\quad,\quad]$ is the commutator.

Equations (4.3.9) coincide with (4.1.18), (4.1.24), whose general solution after being substituted into the equality $F(\psi^*,\, \psi) = H(\psi^*,\, \psi)\psi$ gives rise to formula (4.3.8).

On applying the Lie method we come to the conclusion that the necessary and sufficient conditions for system (4.3.7), (4.3.8) to be invariant under the

group of projective transformations (4.1.13) are as follows

$$(w_1\partial_{w_1} + w_2\partial_{w_2} - 2/3)f_i = 0, \quad i = 1, 2,$$
$$(\gamma_0 + \gamma_4)\Big(i\gamma_a\partial_a + m(\gamma_0 - \gamma_4)\Big)\psi = 0.$$

Since the last equation is not a consequence of system (4.3.7), (4.3.8), equation (4.3.7) is not invariant with respect to the generalized Galilei group having generators (4.3.1). The theorem is proved. $\rhd$

According to the above theorem, to obtain a $G_2(1,3)$-invariant nonlinear generalization of system (4.3.2) we have to study the wider class of PDEs

$$(4im\partial_t - \partial_a\partial_a)\psi - F(\psi^*, \psi, \underset{1}{\psi^*}, \underset{1}{\psi}) = 0, \tag{4.3.10}$$

where the notation $\underset{1}{\psi} = \{\partial_t\psi, \partial_a\psi\}$ is used.

Here we adduce only one example of the equation of the form (4.3.10) invariant under the group $G_2(1,3)$ with generators (4.1.3)

$$\begin{aligned}&(4im\partial_t - \partial_a\partial_a)\psi + (1/3)(\bar{\psi}\psi)^{-1}\Big\{\Big(i(\gamma_0 + \gamma_4)\partial_t - i\gamma_a\partial_a\Big)\bar{\psi}\psi\Big\}\\ &\quad\times\Big\{-i(\gamma_0 + \gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0 - \gamma_4)\Big\}\psi\\ &\quad+(\bar{\psi}\psi)^{2/3}\Big(f_1 + f_2(\gamma_0 + \gamma_4)\Big)\psi = 0,\end{aligned}$$

where $f_i = f_i[(\psi^\dagger\psi + \bar{\gamma}_4\psi)^3(\bar{\psi}\psi)^{-2}]$, $i = 1, 2$ are arbitrary smooth functions.

There exist second-order PDEs invariant under the Galilei group which are principally different from (4.3.2). For example, in [114, 119] the following $G(1,3)$-invariant system of PDEs

$$(i\gamma_\mu\partial_\mu - m)\psi(t, \vec{x}) = (1/2m)(1 - \gamma_0 - i\gamma_4)\partial_a\partial_a\psi(t, \vec{x})$$

was obtained. It is invariant under the Galilei group with the generators

$$\begin{aligned}&P_0 = \partial_t, \quad P_a = \partial_a,\\ &J_{ab} = -x_a\partial_b + x_b\partial_a + (1/2)\gamma_a\gamma_b, \quad a \neq b,\\ &G_a = t\partial_a - imx_a + (1/2)(1 + i\gamma_4)\gamma_a.\end{aligned}$$

SEPARATION OF VARIABLES

In this Chapter we present the basis of the symmetry approach to the separation of variables in systems of linear PDEs. A generalization of the Stäckel method of separation of variables [75, 268] for the case of systems of differential equations is suggested. Separation of variables in some Galilei-invariant PDEs is performed.

5.1. Separation of variables and symmetry of systems of partial differential equations

The Dirac equation (1.1.1) is called separable in Cartesian coordinates if it has exact solutions of the form

$$\psi(x) = V_0(x_0)V_1(x_1)V_2(x_2)V(x_3)\chi, \tag{5.1.1}$$

where V_μ are nonsingular (4×4)-matrices, χ is a constant four-component column.

It is well-known that there exists a deep relation between variable separation and symmetry properties of PDEs [149, 201, 226]. This relation can be characterized in the following way: a solution with separated variables is a common eigenfunction of some set of commuting symmetry operators of the equation considered. To demonstrate the main steps of applying the method of separation of variables within the framework of the symmetry approach we will consider an example. A particular solution of (1.1.1) is looked for as a

solution of the following over-determined system of PDEs:

$$
\begin{aligned}
&(i\gamma_\mu\partial_\mu - m)\psi(x) = 0, \quad (\partial_0 - \lambda_1)\psi(x) = 0,\\
&(\partial_1 - \lambda_2)\psi(x) = 0, \quad (\partial_2 - \lambda_3)\psi(x) = 0.
\end{aligned} \tag{5.1.2}
$$

Integration of the last three equations of system (5.1.2) yields

$$
\psi(x) = \exp\{\lambda_1 x_0 + \lambda_2 x_1 + \lambda_3 x_2\}\varphi(x_3).
$$

Substitution of the above expression into the first equation from (5.1.2) gives rise to the system of ODEs for the four-component function $\varphi(\omega)$

$$
i\gamma_3\dot{\varphi} + (-m + i\lambda_1\gamma_0 + i\lambda_2\gamma_1 + i\lambda_3\gamma_2)\varphi = 0,
$$

whose general solution reads

$$
\varphi(\omega) = \exp\{i\gamma_3(-m + i\lambda_1\gamma_0 + i\lambda_2\gamma_1 + i\lambda_3\gamma_2)\omega\}\chi.
$$

Hence we conclude that the general solution of (5.1.2) is of the form (5.1.1)

$$
\begin{aligned}
\psi(x) \quad = \quad & \exp\{\lambda_1 x_0\}\exp\{\lambda_2 x_1\}\exp\{\lambda_3 x_2\}\\
& \times\exp\{i\gamma_3(-m + i\lambda_1\gamma_0 + i\lambda_2\gamma_1 + i\lambda_3\gamma_2)x_3\}\chi.
\end{aligned} \tag{5.1.3}
$$

Comparing (5.1.1) with (5.1.3) we come to a conclusion that the solution with separated variables (in Cartesian coordinates) is the eigenfunction of operators $\partial_0,\ \partial_1,\ \partial_2$ which form a commutative subalgebra of the invariance algebra of the Dirac equation. Consequently, classification of commuting symmetry operators is a part of the method of separation of variables.

From the above example it is seen that the solution with separated variables contains arbitrary parameters λ_1, λ_2, λ_3 which are called separation constants.

Now we turn to the problem of variable separation in arbitrary systems of linear first-order PDEs

$$
\{L_\mu(x)\partial_\mu + M(x)\}u(x) = 0, \tag{5.1.4}
$$

where $u = \Big(u^0(x),\, u^1(x), \ldots, u^{m-1}(x)\Big)^T$; $x = (x_0,\, x_1, \ldots, x_{n-1})$; $\{n, m\} \subset \mathbb{N}$; $L_\mu,\ M$ are $(m \times m)$-matrices (M is supposed to be nonsingular).

In what follows, a block $(n_1N_1 \times n_2N_2)$-matrix B, whose entries are $(N_1 \times N_2)$-matrices $B_{\mu\nu},\ \mu = 1, \ldots, n_1,\ \nu = 1, \ldots, n_2$, is designated for brevity as

$B = \|B_{\mu\nu}\|_{\mu=1}^{n_1}{}_{\nu=1}^{n_2}$. Such a notation is very convenient and simplifies considerably all manipulations with block matrices. For example, a product of two block $(n_1N_1 \times n_2N_2)$- and $(n_2N_2 \times n_3N_3)$-matrices

$$B = \|B_{\mu\nu}\|_{\mu=1}^{n_1}{}_{\nu=1}^{n_2}, \quad C = \|C_{\mu\nu}\|_{\mu=1}^{n_2}{}_{\nu=1}^{n_3}$$

is a block $(n_1N_1 \times n_3N_3)$-matrix

$$BC = \|B_{\mu\alpha}C_{\alpha\nu}\|_{\mu=1}^{n_1}{}_{\nu=1}^{n_3},$$

where summation over the repeated indices from 1 to n_2 is understood. More details about operations with block matrices can be found in [173].

In the theory of variable separation in linear PDEs with one dependent variable a very important role is played by the Stäckel matrices $C = \|c_{\mu\nu}\|$, $\det C \neq 0$, where $c_{\mu\nu}$ are smooth functions depending on the variable x_μ only. Separable PDEs admit rather natural and simple description in terms of the Stäckel matrices [201]. It is believed that the above matrices when properly generalized should be of importance for variable separation in multi-component systems of linear PDEs as well [227].

Below we present an approach to variable separation in systems of linear PDEs (5.1.4) which uses essentially a generalized block Stäckel matrix introduced below.

Definition 5.1.1. Block $(nm \times nm)$-matrix $C = \|C_{\mu\nu}(x_\mu)\|_{\mu,\nu=0}^{n-1}$, where $C_{\mu\nu}(x_\mu)$ are square $(m \times m)$ matrices, is called the Stäckel matrix if the following conditions are fulfilled:

$$\begin{aligned} &1) \quad \det C \neq 0, \\ &2) \quad [C_{\mu\nu}, C_{\alpha\beta}] + [C_{\mu\beta}, C_{\alpha\nu}] = 0. \end{aligned}$$

Evidently, provided $m = 1$, the above definition coincides with the usual definition of the Stäckel matrix (see e.g. [201, 227]).

Definition 5.1.2. A set of smooth real-valued functions $z_\mu = z_\mu(x)$, $\mu = 0, \ldots, n-1$ is called a coordinate system if the condition $\det \|\partial_{x_\mu} z_\nu(x)\|_{\mu,\nu=0}^{n-1} \neq 0$ is satisfied.

Now we are ready to give a precise definition of separation of variables in systems of linear PDEs which has been suggested for the first time in [169].

Definition 5.1.3. Let $(m \times m)$-matrix functions $V_\mu(z_\mu)$, $\mu = 0, \ldots, n-1$

satisfy the system of n matrix ODEs

$$\begin{aligned} \frac{dV_\mu}{dz_\mu} &= \Big(C_{\mu 0}(z_\mu) + C_{\mu a}(z_\mu)\lambda_a\Big)V_\mu, \quad \mu = 0, \ldots, n-1, \\ V_\mu \Bigg|_{\substack{z_\mu = \theta_\mu \\ \lambda_a = 0}} &= I, \end{aligned} \tag{5.1.5}$$

where $C = \|C_{\mu\nu}(z_\mu)\|_{\mu,\nu=0}^{n-1}$ is a Stäckel matrix, $\lambda_1, \ldots, \lambda_{n-1}$ are arbitrary parameters taking values in some open domain $\Lambda \subset \mathbb{R}^{n-1}$, I is the unit $(m \times m)$-matrix, θ_μ are arbitrary fixed real constants. We say that the system of linear PDEs (5.1.4) is separable in the coordinate system $z_0(x), \ z_1(x), \ldots, z_{n-1}(x)$ if there exist such a $(m \times m)$-matrix $A(x)$ and such a Stäckel matrix C that substitution of the Ansatz

$$u(x) = A(x) \prod_{\mu=0}^{n-1} V_\mu(z_\mu, \vec{\lambda})\chi, \tag{5.1.6}$$

where $V_\mu(z_\mu), \ \mu = 0, \ldots, n-1$ are solutions of system of ODEs (5.1.5) and χ is an arbitrary m-component constant column, into (5.1.4) yields an identity with respect to $\vec{\lambda} \in \Lambda$.

Our aim is to solve the following mutually related problems:

- to describe separable systems of PDEs (5.1.4) in terms of the corresponding Stäckel matrices,
- to establish a correspondence between separability of systems of PDEs and their symmetry properties.

Solution of the first problem is necessary for general understanding of the mechanism of variable separation in systems of linear PDEs and for classification of separable systems. Solving the second problem we obtain a practical tool for finding coordinate systems providing variable separation in a given system of linear PDEs.

Before adducing the principal assertions we make an important remark. It is readily seen that if a system of linear ODEs (5.1.4) is separable in a coordinate system $z_\mu = z_\mu(x)$, then the equation

$$\{L'_\mu(z)\partial_{z_\mu} + M'(x)\}w(z) = 0,$$

obtained from (5.1.4) by means of the change of variables

$$z_\mu = z_\mu(x), \quad w(z) = A^{-1}(x)u(x),$$

is separable in the coordinate system $z'_\mu = z_\mu$ and what is more, the solution with separated variables (5.1.6) reads

$$w(z) = \prod_{\mu=0}^{n-1} V_\mu(z_\mu, \vec{\lambda})\chi.$$

Consequently, when classifying separable systems we can consider separation in Cartesian coordinates $z_\mu = x_\mu$ only and choose $A(x) = I$. With this remark the solution with separated variables (5.1.6) takes the form

$$u(x) = \prod_{\mu=0}^{n-1} V_\mu(x_\mu, \vec{\lambda})\chi. \tag{5.1.7}$$

Theorem 5.1.1. *Equation (5.1.4) is separable iff there exists a Stäckel matrix C satisfying the condition*

$$L_\mu(x)C_{\mu\nu}(x_\mu) = -\delta_{\nu 0}M(x). \tag{5.1.8}$$

Proof. The necessity. Let system of PDEs (5.1.4) be separable. Then, according to Definition 5.1.3 there is such a Stäckel matrix C that solutions $V_\mu(x_\mu)$ of the matrix system of ODEs

$$\begin{aligned} &\frac{dV_\mu}{dx_\mu} = \Big(C_{\mu 0}(x_\mu) + C_{\mu a}(x_\mu)\lambda_a\Big)V_\mu, \quad \mu = 0, \dots, n-1,\\ &V_\mu \Bigg|_{\substack{x_\mu = \theta_\mu \\ \lambda_a = 0}} = I, \end{aligned} \tag{5.1.9}$$

after being substituted into (5.1.7) give rise to an exact solution of the initial system of PDEs (5.1.4) with an arbitrary $\vec{\lambda} \in \Lambda$.

Inserting (5.1.7) into (5.1.4) with account of (5.1.9) we get

$$\begin{aligned} &L_0(C_{00} + C_{0a}\lambda_a\Big)V_1V_2 \cdot \dots \cdot V_{n-1}\chi\\ &\quad + L_1V_0(C_{10} + C_{1a}\lambda_a\Big)V_2V_3 \cdot \dots \cdot V_{n-1}\chi + \dots\\ &\quad + L_{n-1}V_0V_1 \cdot \dots \cdot V_{n-2}(C_{n-10} + C_{n-1a}\lambda_a\Big)\chi + M\chi = 0. \end{aligned} \tag{5.1.10}$$

Using properties of the Stäckel matrix C it is not difficult to prove that the matrices $A_\mu(x_\mu) = C_{\mu 0}(x_\mu) + C_{\mu a}(x_\mu)\lambda_a$, $\mu = 0, \dots, n-1$ are mutually commuting.

Indeed,

$$\begin{aligned}[A_\mu,\, A_\nu] &= [C_{\mu 0},\, C_{\nu 0}] + \lambda_a\Big([C_{\mu 0},\, C_{\nu a}] + [C_{\mu a},\, C_{\nu 0}]\Big)\\ &\quad +\lambda_a\lambda_b\Big([C_{\mu a},\, C_{\nu b}] + [C_{\mu b},\, C_{\nu a}]\Big) = 0.\end{aligned}$$

Since $V_\mu(x_\mu)$ are solutions of the Cauchy problem (5.1.9), they can be represented by the following converging series [197]:

$$V_\mu = I + \int\limits_{\theta_\mu}^{x_\mu} A_\mu(\tau)d\tau + \int\limits_{\theta_\mu}^{x_\mu} A_\mu(\tau)\int\limits_{\theta_\mu}^{\tau} A_\mu(\tau_1)d\tau_1 d\tau + \dots, \quad \mu = 0, \dots, n-1.$$

Hence, it follows that $[A_\mu,\, V_\nu] = 0$ under $\mu \neq \nu$. With this fact relation (5.1.10) is rewritten in the form

$$\Big(L_\mu(C_{\mu 0} + C_{\mu a}\lambda_a) + M\Big)\prod_{\mu=0}^{n-1} V_\mu \chi = 0. \tag{5.1.11}$$

Since χ is an arbitrary m-component constant column and matrices V_μ are invertible, the above equality is equivalent to the following one:

$$L_\mu(C_{\mu 0} + C_{\mu a}\lambda_a) + M = 0.$$

Splitting the equality obtained with respect to λ_a we arrive at the conditions (5.1.8).

The sufficiency. Let V_μ be solutions of (5.1.9) with a Stäckel matrix C satisfying (5.1.8). Inserting the Ansatz (5.1.7) into (5.1.4) and taking into account the relations $[A_\mu,\, V_\nu] = 0,\ \mu \neq \nu$ we get the equality (5.1.11). Hence it follows that the function (5.1.7) satisfies the initial system of PDEs (5.1.4) identically with respect to $\vec{\lambda} \in \Lambda$. The theorem is proved. $\rhd$

Let $B = \|B_{\mu\nu}(x)\|_{\mu,\nu=0}^{n-1}$ be the inverse of the Stäckel matrix $C = \|C_{\mu\nu}(x_\mu)\|_{\mu,\nu=0}^{n-1}$, i.e.,

$$B_{\mu\alpha}C_{\alpha\nu} = C_{\mu\alpha}B_{\alpha\nu} = \delta_{\mu\nu}I.$$

Then, multiplying (5.1.8) by $B_{\nu\mu}$ on the right and summing over ν we arrive at the following representation of the matrices L_μ:

$$L_\mu = -MB_{0\mu}, \quad \mu = 0, \dots, n-1. \tag{5.1.12}$$

Consequently, Theorem 5.1.1 admits an equivalent formulation: *the system of linear PDEs (5.1.4) is separable iff the matrix coefficients L_μ, M are given in the Stäckel form (5.1.12).* Thus, we have proved an analogue of the well-known theorem about variable separation in PDEs with one dependent variable [201, 227].

Theorem 5.1.1 provides a description of separable systems of PDEs via the corresponding Stäckel matrices but it gives no method for construction of solutions with separated variables for specific equations. As stated above, the most effective method for separating variables in systems of linear PDEs is utilization of their symmetry properties. We will show that our definition of variable separation in a system of PDEs is consistent with its symmetry properties. Furthermore, we will obtain a simple description of a solution with separated variables in terms of the first-order symmetry operators of the system under consideration

First,we will prove an auxiliary lemma.

Lemma 5.1.2. *Let $\|B_{\mu\nu}(x)\|^{n-1}_{\mu,\nu=0}$ be a block nonsingular $(nm \times nm)$-matrix. The inverse of it is designated as $\|H_{\mu\nu}(x)\|^{n-1}_{\mu,\nu=0}$. Then matrix functions $B_{\mu\nu}(x)$, $B_\mu(x)$ satisfy the system of PDEs*

$$\begin{aligned}
&1)\quad [B_{\mu\alpha},\, B_{\nu\beta}] + [B_{\mu\beta},\, B_{\nu\alpha}] = 0,\\
&2)\quad [B_{\mu\alpha},\, B_\nu] - [B_{\nu\alpha},\, B_\mu] + B_{\mu\beta}\partial_\beta B_{\nu\alpha} - B_{\nu\beta}\partial_\beta B_{\mu\alpha} = 0,\\
&3)\quad B_{\mu\alpha}\partial_\alpha B_\nu - B_{\nu\alpha}\partial_\alpha B_\mu + [B_\mu,\, B_\nu] = 0,
\end{aligned} \tag{5.1.13}$$

iff matrix functions $H_{\mu\nu}(x)$, $H_\mu(x) = -H_{\mu\nu}B_\nu$ satisfy the system of PDEs

$$\begin{aligned}
&1)\quad [H_{\mu\alpha},\, H_{\nu\beta}] + [H_{\mu\beta},\, H_{\nu\alpha}] = 0,\\
&2)\quad \partial_\nu H_{\mu\alpha} - \partial_\mu H_{\nu\alpha} + [H_{\mu\alpha},\, H_\nu] - [H_{\nu\alpha},\, H_\mu] = 0,\\
&3)\quad \partial_\nu H_\mu - \partial_\mu H_\nu + [H_\mu,\, H_\nu] = 0.
\end{aligned} \tag{5.1.14}$$

In (5.1.13), (5.1.14) subscripts $\mu,\ \nu,\ \alpha,\ \beta$ take the values $0, 1, 2, \ldots, n-1$.

Proof. Consider an over-determined system of PDEs

$$(B_{\mu\nu}\partial_\nu + B_\mu)u = \lambda_\mu u, \quad \mu = 0, \ldots, n-1. \tag{5.1.15}$$

According to Theorem 1.5.3 the above system is compatible iff conditions

$$[B_{\mu\alpha}\partial_\alpha + B_\mu,\, B_{\nu\beta}\partial_\beta + B_\nu] = 0 \tag{5.1.16}$$

hold true. Computing commutators in the left-hand sides of (5.1.16) and equating to zero coefficients of the linearly independent operators $\partial_\mu\partial_\nu,\ \partial_\mu,\ I$ we get equations (5.1.13). Consequently, the system of PDEs (5.1.13) provides the necessary and sufficient compatibility conditions for system (5.1.15).

Next, multiplying both parts of (5.1.15) by $H_{\alpha\mu}$ on the left and summing over μ we have

$$\partial_\mu u = H_{\mu\nu}(\lambda_\nu - B_\nu)u. \tag{5.1.17}$$

The compatibility criterion $\partial_\mu(\partial_\nu u) = \partial_\nu(\partial_\mu u)$ for the system (5.1.17) yields the identities

$$\partial_\mu\Big(H_{\nu\alpha}(\lambda_\alpha - B_\alpha)u\Big) = \partial_\nu\Big(H_{\mu\alpha}(\lambda_\alpha - B_\alpha)u\Big),$$

whence it follows that $(m \times m)$-matrices $H_{\mu\nu}(x),\ H_\mu(x) = -H_{\mu\nu}B_\nu$ satisfy the system of PDEs (5.1.14). The lemma is proved. $\rhd$

Theorem 5.1.2. *Let the system of PDEs (5.1.4) be separable. Then, a solution with separated variables $u(x)$ is a common eigenfunction of commuting first-order differential operators $Q_1,\ Q_2,\dots,Q_{n-1}$ which are symmetry operators of system (5.1.4).*

Proof. As earlier, we designate by the symbol $B = \|B_{\mu\nu}(x)\|_{\mu,\nu=0}^{n-1}$ the inverse of the Stäckel matrix $C = \|C_{\mu\nu}(x_\mu)\|_{\mu,\nu=0}^{n-1}$. Due to the properties of the Stäckel matrix C, the matrix functions $H_{\mu\nu} = C_{\mu\nu}(x_\mu),\ H_\mu = 0$ satisfy system (5.1.14). Hence it follows (Lemma 5.1.2) that the matrix functions $B_{\mu\nu}(x),\ B_\mu(x) = 0$ satisfy equations (5.1.13). Consequently, the operators $Q_\mu = B_{\mu\nu}\partial_\nu$ commute.

By definition the solution with separated variables $u(x) = \prod_{\mu=0}^{n-1} V_\mu(x_\mu, \vec{\lambda})\chi$ satisfies the system of PDEs

$$\partial_\mu u = \Big(C_{0\mu}(x_\mu) + C_{\mu a}(x_\mu)\lambda_a\Big)u. \tag{5.1.18}$$

Multiplying both parts of (5.1.18) by $B_{\alpha\mu}(x)$ on the left and summing over μ we obtain

$$B_{\alpha\mu}\partial_\mu u = (\delta_{\alpha 0} I + \delta_{\alpha a}\lambda_a)u. \tag{5.1.19}$$

Putting in (5.1.19) $\alpha = 0, 1, 2, \dots, n-1$ we arrive at the relations

$$\begin{aligned} B_{0\mu}\partial_\mu u &= u, \\ B_{a\mu}\partial_\mu u &= \lambda_a u, \quad a = 1, \dots, n-1. \end{aligned} \tag{5.1.20}$$

But according to (5.1.12) $B_{0\mu} = -M^{-1}L_\mu,\ \mu = 0, \ldots, n-1$, whence

$$[B_{a\mu}\partial_\mu,\, M^{-1}L_\mu\partial_\mu + I] = -[B_{a\mu}\partial_\mu,\, B_{0\mu}\partial_\mu] = 0.$$

Now we will show that Q_a are symmetry operators, which will complete the proof. Indeed,

$$\begin{aligned}
&[Q_a,\, L_\mu\partial_\mu + M] \equiv [Q_a,\, M(M^{-1}L_\mu\partial_\mu + I)]\\
&\quad = M[Q_a,\, M^{-1}L_\mu\partial_\mu + I] + [Q_a,\, M](M^{-1}L_\mu\partial_\mu + I)\\
&\quad = (B_{a\mu}\partial_\mu M + B_a M)M^{-1}(L_\mu\partial_\mu + M) \equiv R_a(x)(L_\mu\partial_\mu + M),
\end{aligned}$$

the same which is required. The theorem is proved. $\rhd$

Note 5.1.1. A class of solutions with separated variables of a given system of linear PDEs can be considerably extended if we define these by formula (5.1.6) without imposing additional constraints on the matrix functions $V_\mu(z_\mu, \vec{\lambda})$. A peculiar example is the four-component complex-valued function:

$$\psi(\vec{x}) = \exp\{-i\lambda_1(\gamma_0 + \gamma_4)x_1\} \exp\Big\{-\Big(\lambda_2 + (1/2)\gamma_0\gamma_4\Big) \ln x_2\Big\}\varphi\,(x_2/x_3), \tag{5.1.21}$$

which is a solution with separated variables in the coordinate system $z_0 = x_1,\ z_1 = \ln x_2,\ z_2 = x_2/x_3$ of the spinor equation:

$$\Big(\varepsilon_{abc}\gamma_a\gamma_b\partial_c + m/x_2 + f(x_2/x_3)(\gamma_0 + \gamma_4)\Big)\psi(\vec{x}) = 0, \tag{5.1.22}$$

where $m = \text{const}$, f is an arbitrary real-valued function.

The function (5.1.21) is a “generalized” eigenfunction of the symmetry operators $Q_1 = \partial_1,\ Q_2 = x_1\partial_1 + x_2\partial_2 + (1/2)\gamma_0\gamma_4$ of the system of PDEs (5.1.22) in a sense that it satisfies the following equalities:

$$Q_1\psi = \lambda_1(\gamma_0 + \gamma_4)\psi, \quad Q_2\psi = \lambda_2\psi,$$

and what is more, the operators Q_1 and Q_2 do not commute.

However such a class of solutions with separated variables is too large to be described by means of the classical symmetry of the equation under study. To give a symmetry interpretation of these solutions it is necessary to study *conditional symmetry* of systems of linear PDEs [149, 152]. Unlike the classical case, the determining equations for conditional symmetry operators are nonlinear. By this reason, a systematic description of solutions with separated variables (5.1.6) without imposing additional constraints on the form of functions $V_\mu(z_\mu, \vec{\lambda})$ seems to be impossible.

According to Theorem 5.1.2, a solution with separated variables in the sense of Definition 5.1.3 has to be looked for as an eigenfunction of some commuting symmetry operators of the equation under study. Consequently, we can formulate the following symmetry approach to the problem of variable separation in systems of linear PDEs of the form (5.1.4):

- at the first step, the symmetry properties of (5.1.4) in the class $\mathcal{M}_1$ of the first-order differential operators with matrix coefficients are investigated;
- at the second step, the $(n-1)$-dimensional commutative subalgebras of the symmetry algebra are classified;
- at the third step, a compatible over-determined system of PDEs

$$\begin{aligned} &(L_\mu \partial_\mu + M)u = 0, \\ &Q_a u = (B_{a\mu}(x)\partial_\mu + B_a(x))u = \lambda_a u, \quad a = 1, \dots, n-1, \end{aligned} \tag{5.1.23}$$

where $Q_1,\ Q_2, \dots, Q_{n-1}$ are commuting symmetry operators (Lie or non-Lie ones) of equation (5.1.4), is transformed to a separated form

$$\partial_{z_\mu} w = \Big(C_{\mu 0}(z_\mu) + C_{\mu a}(z_\mu)\lambda_a\Big)w, \tag{5.1.24}$$

by a proper change of variables

$$z_\mu = z_\mu(x), \quad w(z) = A^{-1}(x)u(x). \tag{5.1.25}$$

If it is possible to implement the above three steps, then due to Theorem 5.1.2 the initial system of PDEs (5.1.4) is separable in coordinates $z_\mu = z_\mu(x)$, $\mu = 0, \dots, n-1$ and solution with separated variables has the form (5.1.6), where $V_\mu(z_\mu, \vec{\lambda})$ are $(m \times m)$-matrices satisfying systems of ODEs

$$\frac{dV_\mu}{dz_\mu} = \Big(C_{\mu 0}(z_\mu) + C_{\mu a}(z_\mu)\lambda_a\Big)V_\mu, \quad \mu = 0, \dots, n-1$$

(no summation over μ).

The most difficult problem to be solved in the framework of the above approach is a choice of an appropriate change of variables (5.1.25). A regular method for finding such a change is known only for the case, when operators Q_a are Lie symmetry operators. Otherwise, we have to solve a *nonlinear* problem in order to get an explicit form of the "new" variables $z_\mu = z_\mu(x)$ and the matrix function $A(x)$.

In the next two sections we will apply the approach suggested to some Galilei-invariant PDEs.

5.2. Separation of variables in the Galilei-invariant spinor equation

The problem of variable separation in the Dirac equation (1.1.1) was studied intensively by many researchers [12, 13, 48, 59, 196, 227, 256], a number of important results were obtained. Nevertheless, they did not succeed in creating the complete theory (as it was the case for the Hamilton-Jacobi equation) of variable separation in equation (1.1.1).

Analyzing the methods applied we come to the conclusion that the most effective ones are those based on symmetry properties of the Dirac equation. V.N. Shapovalov and G.G. Ekle in [256] described solutions of the system of PDEs (1.1.1) with separated variables which were eigenfunctions of triplets of mutually commuting first-order symmetry operators (a complete description of such operators is given by Theorem 1.1.3). They have obtained 29 inequivalent ($P(1,3)$ non-conjugate) triplets of mutually commuting first-order symmetry operators, each one giving rise to a solution of the Dirac equation with separated variables.

In addition, we can construct a solution with separated variables by using symmetry operators of the order higher than one. In particular, a number of papers (see [12, 13] and references therein) are devoted to the application of the second-order symmetry operators to variable separation in the Dirac equation.

At the same time, the problem of variable separation in spinor PDEs invariant under the Galilei group has not been studied yet. In the present section we will carry out separation of variables in the system of linear PDEs for the spinor field (4.1.1) by using its Lie and non-Lie symmetry described by Theorems 4.1.1, 4.1.3.

To apply the approach developed in the previous section we have, first of all, to describe inequivalent triplets of mutually commuting symmetry operators of equation (4.1.1). To this end, we need the following assertion.

Theorem 5.2.1. *Let* $Q_1 = Q_1^{(\ell)} + Q_1^{(n)}$, $Q_2 = Q_2^{(\ell)} + Q_2^{(n)}$ *be linear combinations of the first-order symmetry operators of equation (4.1.1) with real coefficients and besides* $Q_1^{(\ell)}$, $Q_2^{(\ell)}$ *be linear combinations of Lie symmetry operators and*

$Q_1^{(n)},\ Q_2^{(n)}$ *be linear combinations of non-Lie ones. Let the operators* $Q_1,\ Q_2$ *commute, then* $C_1Q_1^{(n)} + C_2Q_2^{(n)} = 0$ *with some non-vanishing simultaneously real constants* $C_1,\ C_2$.

Proof. The proof of the assertion demands very involved computations, therefore only a general scheme of it will be given.

We declare the operators $t,\ x_a$ to be of the degree $+1$, the operators $\partial_t,\ \partial_a,\ im$ to be of the degree -1, the operators $I,\ \gamma_\mu$ to be of the degree 0. In addition, we assume that the zero operator 0 has an arbitrary degree. With such assumptions the set of the symmetry operators of equation (4.1.1) separates into the three classes

1) operators of the degree -1

$$P_0, \quad P_a, \quad W_0, \quad W_a, \quad S_a, \quad T_a;$$

2) operators of the degree 0

$$J_{ab}, \quad G_a, \quad D, \quad M_1, \quad M_2, \quad R_0, \quad R_a, \quad N_0, \quad N_a;$$

3) operators of the degree $+1$

$$A, \quad K_a.$$

It is easy to see that the relation

$$[Q(n),\, Q(\ell)] = Q(n+\ell), \tag{5.2.1}$$

where $Q(k)$ is a symmetry operator of the degree k, holds. Representing the operators $Q_1,\ Q_2$ in the form

$$Q_i = Q_i(-1) + Q_i(0) + Q_i(+1), \quad i = 1, 2$$

and using (5.2.1) we get

$$\begin{aligned}
[Q_1,\, Q_2] \quad = \quad & [Q_1(-1),\, Q_2(-1)] + [Q_1(-1),\, Q_2(0)] + [Q_1(0),\, Q_2(-1)] \\
& + [Q_1(+1),\, Q_2(-1)] + [Q_1(0),\, Q_2(+1)] + [Q_1(+1),\, Q_2(0)] \\
& + [Q_1(+1),\, Q_2(+1)] + [Q_1(0),\, Q_2(0)] + [Q_1(-1),\, Q_2(+1)] \\
& = Q(-2) + Q(-1) + Q(0) + Q(+1) + Q(+2) = 0.
\end{aligned}$$

From the above equalities we obtain the following relations:

$$Q(-2) = Q(-1) = Q(0) = Q(+1) = Q(+2) = 0.$$

Consequently, $[Q_1(-1),\, Q_2(-1)] = 0$ or

$$\begin{aligned}&[a_0^{(1)}W_0 + a_a^{(1)}W_a + b_a^{(1)}S_a + c_a^{(1)}T_a + d_0^{(1)}P_0 + d_a^{(1)}P_a,\\ &\quad a_0^{(2)}W_0 + a_a^{(2)}W_a + b_a^{(2)}S_a + c_a^{(2)}T_a + d_0^{(2)}P_0 + d_a^{(2)}P_a] = 0.\end{aligned}$$

Computing the commutator in the left-hand side of the above equality and equating coefficients of linearly independent operators we arrive at the conclusion that there exist such real constants $C_1,\ C_2$ that

$$\begin{aligned}C_1a_0^{(1)} + C_2a_0^{(2)} = 0, \quad C_1a_a^{(1)} + C_2a_a^{(2)} = 0,\\ C_1b_a^{(1)} + C_2b_a^{(2)} = 0, \quad C_1c_a^{(1)} + C_2c_a^{(2)} = 0,\end{aligned} \tag{5.2.2}$$

where $a = 1, 2, 3$, and what is more $C_1^2 + C_2^2 \neq 0$ (without loss of generality we may choose $C_2 \neq 0$).

Due to (5.2.2) the equality

$$0 = Q(-1) = [Q_1(-1),\, Q_2(0)] + [Q_1(0),\, Q_2(-1)]$$

takes the form

$$[Q_1^{(n)}(-1),\, C_1Q_1(0) + C_2Q_2(0)] + [\alpha_0P_0 + \alpha_aP_a,\, Q_1(0)] = 0$$

with some real constants $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3$.

Computing the commutator and equating coefficients of the linearly-independent operators we arrive at the condition

$$C_1Q_1^{(n)}(0) + C_2Q_2^{(n)}(0) = 0.$$

Similarly,

$$C_1Q_1^{(n)}(+1) + C_2Q_2^{(n)}(+1) = 0.$$

Thus, we have established that there exist such non-vanishing simultaneously real numbers $C_1,\ C_2$ that

$$\begin{aligned}C_1Q_1^{(n)} + C_2Q_2^{(n)} \;&=\; C_1\Big(Q_1^{(n)}(-1) + Q_1^{(n)}(0) + Q_1^{(n)}(+1)\Big)\\ &\quad +C_2\Big(Q_2^{(n)}(-1) + Q_2^{(n)}(0) + Q_2^{(n)}(+1)\Big) = 0.\end{aligned}$$

The theorem is proved. $\rhd$

Note 5.2.1 As established in [198, 256] the above assertion holds true for the first-order symmetry operators of the Dirac equation.

Theorem 5.2.1 simplifies substantially the problem of classification of inequivalent triplets of the mutually commuting symmetry operators of equation (4.1.1). Since we look for a solution with separated variables as a solution of over-determined system of PDEs (5.1.23), triplets of the symmetry operators $\langle Q_1, Q_2, Q_3\rangle$ and $\langle Q_1, Q_2, C_1Q_1 + C_2Q_2 + C_3Q_3\rangle$ with $C_3 \neq 0$ are equivalent. Hence, by using Theorem 5.2.1, it follows that triplets of the mutually commuting symmetry operators belong to one of the following classes:

$$\begin{aligned}&\text{I. } \langle Q_1^{(\ell)} + Q_1^{(n)}, Q_2^{(\ell)}, Q_3^{(\ell)}\rangle,\\ &\text{II. } \langle Q_1^{(n)}, Q_2^{(\ell)}, Q_3^{(\ell)}\rangle,\end{aligned}\tag{5.2.3}$$

where we designate by the symbol $Q_a^{(\ell)}$ a linear combination of the Lie symmetry operators and by the symbol $Q_1^{(n)}$ a linear combination of the non-Lie symmetry operators.

By the arguments used while proving Theorem 5.2.1 we establish that the operators $Q_1^{(\ell)} + Q_1^{(n)}$, $Q_2^{(\ell)}$ and $Q_3^{(\ell)}$ commute iff

$$\begin{aligned}&[Q_1^{(\ell)}, Q_2^{(\ell)}] = [Q_1^{(n)}, Q_2^{(\ell)}] = 0,\\ &[Q_1^{(\ell)}, Q_3^{(\ell)}] = [Q_1^{(n)}, Q_3^{(\ell)}] = 0,\\ &[Q_2^{(\ell)}, Q_3^{(\ell)}] = 0.\end{aligned}\tag{5.2.4}$$

Consequently, to classify $G_2(1,3)$ inequivalent triplets of commuting symmetry operators of equation (4.1.1) we can make use of the results of subalgebraic analysis of the Lie algebra of the generalized Galilei group $G_2(1,3)$ which has been carried out in [16, 100].

According to [16, 100] there are 5 three-dimensional and 14 two-dimensional $G_2(1,3)$ non-conjugate commutative subalgebras of the algebra $AG_2(1,3)$. Solving for each of them equations (5.2.4) we get the following assertion.

Theorem 5.2.2. *The list of $G_2(1,3)$ non-conjugate triplets of commuting first-order symmetry operators of equation (4.1.1) is exhausted by the following ones:*

1) $\langle G_1 + \alpha P_0,\ P_2,\ P_3\rangle$,
2) $\langle G_1 + \alpha P_1,\ G_2,\ P_3\rangle$,
3) $\langle G_1 + \alpha P_1,\ G_2 + \beta P_2,\ P_3\rangle$,
4) $\langle J_{12},\ P_0,\ \alpha W_0 + \alpha_1 N_0 + \alpha_2 W_3 + \alpha_3 T_3 + \delta S_3\rangle$,

$$
\begin{array}{rl}
5) & \langle J_{12},\, A+P_0,\, \alpha W_3+\beta N_0+\delta(T_3+K_3)\rangle,\\
6) & \langle J_{12},\, D,\, \alpha W_3+\beta R_3+\delta N_0\rangle,\\
7) & \langle J_{12},\, G_3+P_0,\, \alpha W_3+\beta(R_0+T_3)+\delta S_3\rangle,\\
8) & \langle J_{12},\, P_3,\, \alpha W_0+\alpha_1 W_3+\alpha_2 T_3+\alpha_3 S_3+\delta N_3\rangle,\\
9) & \langle G_1,\, P_2,\, \alpha_a W_a+\beta N_2+\delta S_1\rangle,\\
10) & \langle G_1+P_2,\, P_3,\, \alpha_a W_a+\beta(2W_0-N_3)+\delta S_1\rangle,\\
11) & \langle P_0,\, P_1,\, \alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a\rangle,\\
12) & \langle P_1,\, P_2,\, \alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a\rangle,\\
13) & \langle J_{12}+P_3,\, P_0,\, \alpha W_0+\alpha_1 W_3+\alpha_2 T_3+\delta S_3\rangle,\\
14) & \langle G_3+P_0,\, P_2,\, \alpha_a W_a+\beta(T_1-N_2)+\delta S_3\rangle,\\
15) & \langle G_1+P_2,\, J_{12}+A+P_0,\, \alpha W_3+\beta(T_3+N_0+K_3)\rangle,\\
16) & \langle J_{12}+P_0,\, P_3,\, \alpha W_0+\alpha_1 W_3+\alpha_2 T_3+\alpha_3 S_3\rangle,\\
17) & \langle G_1+P_1+\alpha P_3,\, G_2,\, \alpha_a W_a+\beta[(1+\alpha^2)S_2+N_2\\
 & -\alpha R_0]+\delta(N_1+\alpha N_3)\rangle,\\
18) & \langle G_1+P_1+\alpha P_3,\, G_2+\beta P_3,\, \alpha_3 W_a+\delta(N_1\\
 & -2\beta W_0-\beta^2 S_1-\beta R_0+\alpha\beta S_2+\alpha N_3)\\
 & +\rho[N_2-\beta S_3-\alpha\beta S_1-\alpha R_0+(1+\alpha^2)S_2-\beta N_3]\rangle,\\
19) & \langle P_0+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_1,\, P_2\rangle,\\
20) & \langle P_0,\, P_1+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_2\rangle,\\
21) & \langle P_0+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_2,\, P_3\rangle,\\
22) & \langle G_1+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_2,\, P_3\rangle,\\
23) & \langle G_1,\, P_2+\alpha_a W_a+\beta N_3+\delta S_1,\, P_3\rangle,\\
24) & \langle G_1+P_0+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_2,\, P_3\rangle,\\
25) & \langle G_1+P_0,\, P_2+\alpha_a W_a+\beta(T_2-N_3)+\delta S_1,\, P_3\rangle,\\
26) & \langle G_1+P_1+\alpha_a W_a+\beta N_3+\delta S_2,\, G_2,\, P_3\rangle,\\
27) & \langle G_1+P_1,\, G_2+\alpha_a W_a+\beta S_1,\, P_3\rangle,\\
28) & \langle G_1+P_1,\, G_2,\, P_3+\alpha_a W_a+\beta N_1\rangle,\\
29) & \langle J_{12}+\alpha W_0+\alpha_a W_a+\beta_a T_a+\delta_a S_a,\, P_0,\, P_3\rangle,\\
30) & \langle J_{12},\, P_0+\alpha W_0+\alpha_1 W_3+\alpha_2 T_3+\alpha_3 S_3+\beta N_3,\, P_3\rangle,\\
31) & \langle J_{12},\, P_0,\, P_3+\alpha W_0+\alpha_1 N_0+\alpha_2 W_3+\alpha_3 T_3+\beta S_3\rangle,
\end{array}
\tag{5.2.5}
$$

where $\alpha,\ \alpha_a,\ \beta,\ \beta_a,\ \delta,\ \delta_a,\ \rho$ *are arbitrary real constants.*

We have not succeeded yet in relating each triplet from the list (5.2.5)

to some coordinate system providing variable separation in system of PDEs (4.1.1) (so far it is not clear whether such a relation exists). Another problem is that there exist different triplets yielding the same coordinate system. For example, triplets 8 and 9 from (5.2.5) give rise to solutions of (4.1.1) with separated variables in Cartesian coordinates $t,\ x_a$. In such a case we adduce the most simple triplet of symmetry operators corresponding to a given coordinate system.

We have obtained 16 coordinate systems providing variable separation in equation (4.1.1). As an example, we will consider a procedure of variable separation in the case when all elements of the triplet

$$Q_a = \xi_{a0}(t,\vec{x})\partial_t + \xi_{ab}(t,\vec{x})\partial_b + \eta_a(t,\vec{x}) \tag{5.2.6}$$

belong to the Lie algebra admitted by the equation under study.

Since the above operators commute, there exists such a change of variables [159]

$$\begin{aligned} &z_\mu = z_\mu(t,\vec{x}), \quad \mu = 0,\ldots,3, \\ &\tilde{\psi}(z) = A(t,\vec{x})\psi(t,\vec{x}), \end{aligned} \tag{5.2.7}$$

where $A(t,\ \vec{x})$ is some invertible (4×4)-matrix, that operators (5.2.6) take the form $Q_a = \partial_{z_a}$. And what is more due to Theorem 1.5.1 the initial equation (4.1.1) on the set of solutions of the system of PDEs

$$Q_a\tilde{\psi} = \lambda_a\tilde{\psi} \tag{5.2.8}$$

is rewritten as follows

$$R_0(z_0)\tilde{\psi}_{z_0} + R_1(z_0;\, \lambda_1,\lambda_2,\lambda_3)\tilde{\psi} = 0$$

with some matrices $R_0,\ R_1$.

Thus, the system of PDEs (5.1.19) rewritten in the new variables $z_\mu,\ \tilde{\psi}(z)$ takes the form

$$\begin{aligned} &R_0(z_0)\tilde{\psi}_{z_0} + R_1(z_0;\lambda_1,\lambda_2,\lambda_3)\tilde{\psi} = 0, \\ &\tilde{\psi}_{z_a} = \lambda_a\tilde{\psi}, \quad a = 1,2,3 \end{aligned}$$

i.e., the variables z_μ separate.

On integrating the above systems of ODEs and substituting the result into (5.2.7) we get the solution of equation (4.1.1) with separated variables.

Provided one element of the triplet of symmetry operators is a non-Lie one, there is no general approach to the problem of transforming system (5.1.23) to the "separated" form

$$R_{1\mu}(z_\mu)\tilde{\psi}_{z_\mu} + R_{2\mu}(z_\mu;\, \lambda_1, \lambda_2, \lambda_3)\tilde{\psi} = 0, \quad \mu = 0, \ldots, 3, \tag{5.2.9}$$

(no summation over μ is carried out), where $R_{1\mu}$, $R_{2\mu}$ are some (4×4)-matrices. Each triplet containing non-Lie symmetry operator demands specific and very involved computations.

In the case considered, the problem is a little bit simplified since two elements of the triplet are Lie symmetry operators. Transforming these to the form $Q_a = \partial_{z_a}$, $a = 1, 2$ we get two new variables $z_1(t, \vec{x})$, $z_2(t, \vec{x})$. The third new variable is always $z_0 = t$. So it is necessary to guess the fourth variable $z_3 = z_3(t, \vec{x})$ and the (4×4)-matrix $A(t, \vec{x})$ transforming the system of PDEs (4.1.1) to a separated form (5.2.9). Omitting details of derivation of the corresponding formulae we present the final result: triplets of commuting symmetry operators, coordinate systems providing variable separation and corresponding systems of separated ODEs of the form (5.2.9).

1) $\langle P_0,\, P_1,\, P_2 \rangle$,
$A(t, \vec{x}) = I, \quad z_0 = x_3, \quad z_1 = t, \quad z_2 = x_1, \quad z_3 = x_2,$
$\tilde{\psi}_{z_0} + \{\lambda_1\gamma_3(\gamma_0 + \gamma_4) - \lambda_2\gamma_3\gamma_1 - \lambda_3\gamma_3\gamma_2 + im\gamma_3(\gamma_0 - \gamma_4)\}\tilde{\psi} = 0,$
$\tilde{\psi}_{z_a} = \lambda_a\tilde{\psi}, \quad a = 1, 2, 3;$

2) $\langle J_{12},\, P_0,\, P_3 \rangle$,
$A(t, \vec{x}) = \exp\{-(1/2)z_2\gamma_1\gamma_2\},$
$z_0 = (x_1^2 + x_2^2)^{1/2}, \quad z_1 = t, \quad z_2 = \arctan(x_2/x_1), \quad z_3 = x_3,$
$\tilde{\psi}_{z_0} + \{\lambda_1\gamma_1(\gamma_0 + \gamma_4) - \lambda_2 z_0^{-1}\gamma_1\gamma_2 - \lambda_3\gamma_1\gamma_3$
$\quad + im\gamma_1(\gamma_0 - \gamma_4) + (1/2)z_0^{-1}\}\tilde{\psi} = 0, \quad \tilde{\psi}_{z_a} = \lambda_a\tilde{\psi}, \quad a = 1, 2, 3;$

3) $\langle G_1 + \alpha P_0,\, P_2,\, P_3 \rangle$,
$A(t, \vec{x}) = \exp\{2imz_0z_1 + (i/3)\alpha m z_1^3 + (1/2)z_1\gamma_1(\gamma_0 + \gamma_4)\},$
$z_0 = x_1 - t^2/2\alpha, \quad z_1 = t/\alpha, \quad z_2 = x_2, \quad z_3 = x_3,$
$\tilde{\psi}_{z_0} + \{\alpha^{-1}(\lambda_1 - 2imz_0)\gamma_1(\gamma_0 + \gamma_4) - \lambda_2\gamma_1\gamma_2 - \lambda_3\gamma_1\gamma_3$
$\quad + im\gamma_1(\gamma_0 + \gamma_4)\}\psi = 0, \quad \tilde{\psi}_{z_a} = \lambda_a\tilde{\psi}, \quad a = 1, 2, 3;$

4) $\langle G_1 + \alpha P_1,\, G_2,\, P_3 \rangle$,
$A(t, \vec{x}) = \exp\{imz_0z_2^2 + (1/2)z_2(\gamma_0 + \gamma_4)\gamma_2 + (1/2)z_1(\gamma_0 + \gamma_4)\gamma_1$

$$+im(z_0+\alpha)z_1^2\},$$
$$z_0=t, \quad z_1=x_1/(t+\alpha), \quad z_2=x_2/t, \quad z_3=x_3,$$
$$-i(\gamma_0+\gamma_4)\tilde{\psi}_{z_0}+\{-(1/2)(z_0+\alpha)^{-1}(\gamma_0+\gamma_4)+(i/2)z_0^{-1}(\gamma_0+\gamma_4)$$
$$+i\lambda_1(z_0+\alpha)^{-1}\gamma_1+i\lambda_2 z_0^{-1}\gamma_2+i\lambda_3\gamma_3+m(\gamma_0-\gamma_4)\}\tilde{\psi}=0,$$
$$\tilde{\psi}_{z_a}=\lambda_a\tilde{\psi}, \quad a=1,2,3;$$

5) $\langle G_1+\alpha P_1,\, G_2+\beta P_2,\, G_3\rangle$,
$$A(t,\,\vec{x})=\exp\{im[(z_0+\alpha)z_1^2+(z_0+\beta)z_2^2+z_0z_3^2]$$
$$+(1/2)(\gamma_0+\gamma_4)\gamma_a z_a\},$$
$$z_0=t, \quad z_1=x_1/(t+\alpha), \quad z_2=x_2/(t+\beta), \quad z_3=x_3/t,$$
$$-i(\gamma_0+\gamma_4)\tilde{\psi}_{z_0}+\Big\{i\lambda_1(z_0+\alpha)^{-1}\gamma_1+i\lambda_2(z_0+\beta)^{-1}\gamma_2$$
$$+i\lambda_3 z_0^{-1}\gamma_3+(i/2)\Big((z_0+\alpha)^{-1}+(z_0+\beta)^{-1}+z_0^{-1}\Big)(\gamma_0+\gamma_4)$$
$$+m(\gamma_0-\gamma_4)\Big\}\tilde{\psi}=0, \quad \tilde{\psi}_{z_a}=\lambda_a\tilde{\psi}, \quad a=1,2,3;$$

6) $\langle N_0+\alpha W_0,\, J_{12},\, P_0\rangle$,
$$A(t,\,\vec{x})=\exp\{-(1/2)\gamma_1\gamma_3 z_3\}\exp\{-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0=t, \quad z_1=(x_1^2+x_2^2+x_3^2)^{1/2}, \quad z_2=\arctan(x_2/x_1),$$
$$z_3=\arctan[x_3(x_1^2+x_2^2)^{-1/2}],$$
$$\tilde{\psi}_{z_0}=\lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_2}=-\lambda_2\tilde{\psi},$$
$$\tilde{\psi}_{z_1}=\Big\{\Big((\alpha/2)z_1^{-1}-\gamma_1\Big)\Big(\lambda_1(\gamma_0+\gamma_4)+im(\gamma_0-\gamma_4)\Big)$$
$$+\lambda_3 z_1^{-1}\gamma_0\gamma_4-z_1^{-1}\Big\}\tilde{\psi},$$
$$\tilde{\psi}_{z_3}=\Big\{(1/2)\tan z_3+\lambda_2(\cos z_3)^{-1}\gamma_2\gamma_3-(\alpha/2)\Big(\lambda_1(\gamma_0+\gamma_4)$$
$$-im(\gamma_0-\gamma_4)\Big)\gamma_2-\lambda_3\gamma_2\Big\}\tilde{\psi};$$

7) $\langle N_3+\alpha W_3,\, J_{12},\, P_3\rangle$,
$$A(t,\,\vec{x})=\exp\{imz_0z_1^2+(1/2)(\gamma_0+\gamma_4)\gamma_1 z_1\}\exp\{-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0=t, \quad z_1=(x_1^2+x_2^2)^{1/2}/t, \quad z_2=\arctan(x_2/x_1), \quad z_3=x_3,$$
$$z_0(\gamma_0+\gamma_4)\gamma_3\tilde{\psi}_{z_0}+\Big\{(\gamma_0+\gamma_4)\gamma_3+\lambda_3 z_0\gamma_0\gamma_4+imz_0\gamma_3(\gamma_0-\gamma_4)$$
$$+(\alpha/2)\Big(\lambda_3(\gamma_0+\gamma_4)-2im\gamma_3\Big)-\lambda_1\Big\}\tilde{\psi}=0,$$
$$\gamma_2\tilde{\psi}_{z_1}+\{(1/2)z_1^{-1}\gamma_1\gamma_2+\lambda_3 z_1^{-1}\gamma_1\}\tilde{\psi}=0,$$

$\tilde\psi_{z_2} = -\lambda_2\tilde\psi, \quad \tilde\psi_{z_3} = \lambda_3\tilde\psi;$

8) $\langle G_1,\, P_2,\, N_2 + \varepsilon S_1\rangle, \quad \varepsilon = \pm 1,$

$$A(t,\vec x) = \exp\{-\varepsilon(\gamma_0+\gamma_4)z_3(1+z_0^2)^{-1/2}\}\exp\{imz_0z_3^2 + (1/2)z_0z_3 \times(1+z_0^2)^{-1/2}(\gamma_0+\gamma_4)\gamma_3\}\exp\{-(\varepsilon/2)\gamma_1\gamma_2\arctan z_0\} \times\exp\{imz_0z_1^2 + (1/2)z_1(\gamma_0+\gamma_4)\gamma_1\},$$

$$z_0 = t, \quad z_1 = x_1/t, \quad z_2 = x_2, \quad z_3 = x_3(1+t^2)^{-1/2},$$

$$(1+z_0^2)^{1/2}(\gamma_0+\gamma_4)\gamma_1\tilde\psi_{z_0} + \{(1+z_0^2)^{1/2}(2z_0)^{-1}(\gamma_0+\gamma_4)\gamma_1 +(\varepsilon\lambda_1z_0^{-1}+\lambda_2z_0)\gamma_0\gamma_4 - im(1+z_0^2)^{1/2}(\gamma_0-\gamma_4)\gamma_1 - \lambda_3\}\tilde\psi = 0,$$

$$\tilde\psi_{z_1} = -\lambda_1\tilde\psi, \quad \tilde\psi_{z_2} = \lambda_2\tilde\psi,$$

$$\tilde\psi_{z_3} + \{2imz_3\gamma_3 - (\lambda_1 - \varepsilon\lambda_2)\gamma_2\gamma_3 - \lambda_3\gamma_2\}\tilde\psi = 0;$$

9) $\langle D,\, J_{12},\, N_0\rangle,$

$$A(t,\,\vec x) = \exp\{(1/4)(\gamma_0+\gamma_4)\gamma_1\}\exp\{-(1/2)\gamma_0\gamma_4z_0 + 2\ln z_1 - 2z_0\} \times\exp\{-(1/2)\gamma_1\gamma_3z_3\}\exp\{-(1/2)\gamma_1\gamma_2z_2\},$$

$$z_0 = (1/2)\ln t, \quad z_1 = (x_1^2+x_2^2+x_3^2)^{1/2}t^{-1/2},$$

$$z_2 = \arctan(x_2/x_1), \quad z_3 = \arctan[x_3(x_1^2+x_2^2)^{-1/2}],$$

$$\tilde\psi_{z_0} = \lambda_1\tilde\psi, \quad \tilde\psi_{z_2} = -\lambda_2\tilde\psi,$$

$$\tilde\psi_{z_1} = \{-(1/4)(\gamma_0+\gamma_4)\gamma_1(1+2\lambda_1) - im(\gamma_0-\gamma_4)\gamma_1 + im +(im/4)\gamma_1(\gamma_0+\gamma_4) + \lambda_3z_1^{-1}\gamma_4\gamma_0\}\tilde\psi,$$

$$\tilde\psi_{z_3} = \{\lambda_3\gamma_2 + (1/2)\tan z_3 + \lambda_2(\cos z_3)\gamma_2\gamma_3\}\tilde\psi;$$

10) $\langle D,\, J_{12},\, W_3\rangle,$

$$A(t,\,\vec x) = \exp\{-(1/2)\gamma_0\gamma_4z_0\}\exp\{-(1/2)\gamma_1\gamma_2z_2\},$$

$$z_0 = (1/2)\ln t, \quad z_1 = (x_1^2+x_2^2)^{1/2}t^{-1/2}, \quad z_2 = \arctan(x_2/x_1),$$

$$z_3 = x_3t^{-1/2},$$

$$\tilde\psi_{z_0} = \lambda_1\tilde\psi, \quad \tilde\psi_{z_2} = -\lambda_2\tilde\psi,$$

$$\gamma_1(\gamma_0+\gamma_4)\tilde\psi_{z_1} + \{(1/2)z_1^{-1}\gamma_1(\gamma_0+\gamma_4) - \lambda_2z_1^{-1}\gamma_2(\gamma_0+\gamma_4) -2\lambda_3\gamma_3 - 2im\gamma_0\gamma_4\}\tilde\psi = 0,$$

$$(\gamma_0+\gamma_4)\tilde\psi_{z_3} - \{2im\gamma_3 + 2\lambda_3\}\tilde\psi = 0;$$

11) $\langle A + P_0,\, J_{12},\, N_0\rangle,$

$$A(t,\,\vec x) = \exp\{imz_1^2\tan z_0 - 2\ln(\cos z_0)\}\Big\{\exp\{(1/2)\gamma_0\gamma_4$$

$$\times \ln(\cos z_0)\} + (1/2)z_1 \sin z_0(\cos z_0)^{-1/2}(\gamma_0 + \gamma_4)\gamma_1\Big\}$$
$$\times \exp\{-(1/2)\gamma_1\gamma_3 z_3\} \exp\{-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0 = \arctan t, \quad z_1 = (x_1^2 + x_2^2 + x_3^2)^{1/2}(1 + t^2)^{-1/2},$$
$$z_2 = \arctan(x_2/x_1) \quad , z_3 = \arctan[x_3(x_1^2 + x_2^2)^{-1/2}],$$
$$\tilde{\psi}_{z_0} = \lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_2} = -\lambda_2\tilde{\psi},$$
$$\tilde{\psi}_{z_1} = \Big\{-z_1^{-1} + \lambda_1(\gamma_0 + \gamma_4)\gamma_1 - im\gamma_1(\gamma_0 - \gamma_4) + \lambda_1 z_1^{-1}\gamma_0\gamma_4\Big\}\tilde{\psi},$$
$$\tilde{\psi}_{z_3} = \{(1/2)\tan z_3 - \lambda_2(\cos z_3)^{-1}\gamma_2\gamma_3 + \lambda_3\gamma_2\}\tilde{\psi};$$

12) $\langle A + P_0,\, J_{12},\, W_3\rangle$,
$$A(t,\, \vec{x}) = \exp\{im(z_1^2 + z_3^2)\tan z_0 - 2\ln(\cos z_0)\}\Big\{\exp\{(1/2)\gamma_0\gamma_4$$
$$\times \ln(\cos z_0)\} + (1/2)(\gamma_1 z_1 + \gamma_3 z_3)\sin z_0(\cos z_0)^{-1/2}\Big\}$$
$$\times \exp\{-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0 = \arctan t, \quad z_1 = (x_1^2 + x_2^2)^{1/2}(1 + t^2)^{-1/2},$$
$$z_2 = \arctan(x_2/x_1), \quad z_3 = x_3,$$
$$\tilde{\psi}_{z_0} = \lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_2} = -\lambda_2\tilde{\psi},$$
$$(\gamma_0 + \gamma_4)\gamma_2\tilde{\psi}_{z_1} + \{(2z_1)^{-1}(\gamma_0 + \gamma_4)\gamma_2 + \lambda_2 z_1^{-1}(\gamma_0 + \gamma_4)\gamma_1$$
$$+2(\lambda_3 - im\gamma_1\gamma_2)\}\tilde{\psi} = 0,$$
$$(1/2)(\gamma_0 + \gamma_4)\tilde{\psi}_{z_3} - \{\lambda_3 + im\gamma_3\}\tilde{\psi} = 0;$$

13) $\langle G_3 + P_0,\, J_{12},\, S_3\rangle$,
$$A(t,\, \vec{x}) = \exp\{2im[z_0 z_3 + (1/6)z_0^3] + (1/2)z_0(\gamma_0 + \gamma_4)\gamma_3$$
$$-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0 = t, \quad z_1 = (x_1^2 + x_2^2)^{1/2}, \quad z_2 = \arctan(x_2/x_1),$$
$$z_3 = x_3 - (1/2)t^2,$$
$$\tilde{\psi}_{z_0} = \lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_2} = -\lambda_2\tilde{\psi},$$
$$\tilde{\psi}_{z_1} = \{(2z_1)^{-1} - \lambda_2 z_1^{-1}\gamma_1\gamma_2 + \lambda_3\gamma_2\}\tilde{\psi},$$
$$\tilde{\psi}_{z_3} = \{\lambda_1(\gamma_0 + \gamma_4)\gamma_3 - im\gamma_3(\gamma_0 - \gamma_4) - 2imz_3(\gamma_0 + \gamma_4)\gamma_3$$
$$+\lambda_3\gamma_0\gamma_4\})\tilde{\psi};$$

14) $\langle J_{12} + P_3,\, P_0,\, S_3\rangle$,
$$A(t,\, \vec{x}) = \exp\{(1/2)\gamma_1\gamma_2(z_3 - z_2)\},$$
$$z_0 = t, \quad z_1 = (x_1^2 + x_2^2)^{1/2}, \quad z_2 = \arctan(x_2/x_1) + x_3, \quad z_3 = x_3,$$

$$\tilde{\psi}_{z_0} = \lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_3} = \lambda_3\tilde{\psi},$$
$$\tilde{\psi}_{z_1} = \{(2z_1)^{-1} - \lambda_2 z_1^{-1}\gamma_1\gamma_2 + \lambda_3\gamma_2\}\tilde{\psi},$$
$$\tilde{\psi}_{z_2} = \{-\lambda_2 + \lambda_3\gamma_0\gamma_4 - \lambda_1(\gamma_0+\gamma_4)\gamma_3 - im\gamma_3(\gamma_0-\gamma_4)\}\tilde{\psi};$$

15) $\langle J_{12} + P_0,\, P_3,\, W_0\rangle$,
$$A(t,\vec{x}) = \exp\{-(1/2)\gamma_1\gamma_2 z_2\},$$
$$z_0 = t, \quad z_1 = (x_1^2 + x_2^2)^{1/2}, \quad z_2 = t + \arctan(x_2/x_1), \quad z_3 = x_3,$$
$$\tilde{\psi}_{z_0} = \lambda_2\tilde{\psi}, \quad \tilde{\psi}_{z_3} = \lambda_3\tilde{\psi},$$
$$(\gamma_0+\gamma_4)\gamma_1\tilde{\psi}_{z_1} + \Big\{z_1^{-1}\gamma_2\Big(-2\lambda_1 - \lambda_2(\gamma_0+\gamma_4) + im(\gamma_0-\gamma_4)\Big)$$
$$+\Big(-(2z_1)^{-1}\gamma_1 + 2\lambda_1 - \lambda_3\gamma_3\Big)(\gamma_0+\gamma_4)\Big\}\tilde{\psi} = 0,$$
$$(\gamma_0+\gamma_4)\tilde{\psi}_{z_2} + \{2\lambda_1 + \lambda_2(\gamma_0+\gamma_4) - im(\gamma_0-\gamma_4)\}\tilde{\psi} = 0;$$

16) $\langle G_1 + P_2,\, P_3,\, S_1\rangle$,
$$A(t,\vec{x}) = \exp\{imz_1^2 z_0 + (z_1/2)\gamma_1(\gamma_0+\gamma_4)\},$$
$$z_0 = t, \quad z_1 = x_1/t, \quad z_2 = x_2 - x_1/t, \quad z_3 = x_3,$$
$$-i(\gamma_0+\gamma_4)\gamma_1\tilde{\psi}_{z_0} + \{-i(2z_0)^{-1}(\gamma_0+\gamma_4) + i\lambda_1 z_0^{-1}\gamma_1 + i\lambda_2\gamma_2\gamma_3$$
$$+m(\gamma_0-\gamma_4)\}\tilde{\psi} = 0, \quad \tilde{\psi}_{z_1} = \lambda_1\tilde{\psi}, \quad \tilde{\psi}_{z_3} = \lambda_3\tilde{\psi},$$
$$\tilde{\psi}_{z_2} = \{\lambda_2\gamma_3 + \lambda_3\gamma_2\gamma_3\}\tilde{\psi}.$$

In the above formulae α, β are arbitrary real parameters, λ_1, λ_2, λ_3 are separation constants.

Note that coordinate systems given in the formulae 1–5 correspond to the Lie symmetry of the system of PDEs (4.1.1) and the ones given in the formulae 6–16 correspond to its non-Lie symmetry.

5.3. Separation of variables in the Schrödinger equation

The problem of separation of variables in the two-dimensional Schrödinger equation
$$iu_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1,x_2)u \tag{5.3.1}$$
as well as a majority of classical problems of mathematical physics can be formulated in a very simple way (but this simplicity does not, of course, imply existence of an easy way to its solution). To separate variables in equation

(5.3.1) we have to construct such functions $R(t,\vec{x}),\ \omega_1(t,\vec{x}),\ \omega_2(t,\vec{x})$ that the Schrödinger equation (5.3.1) after being rewritten in the new variables

$$\begin{aligned}&z_0 = t,\ z_1 = \omega_1(t,\vec{x}),\ z_2 = \omega_2(t,\vec{x}),\\ &v(z_0,\vec{z}) = R(t,\vec{x})u(t,\vec{x})\end{aligned} \tag{5.3.2}$$

separates into three ordinary differential equations (ODEs) by means of the substitution $v = \varphi_0(z_0)\varphi_1(z_1)\varphi_2(z_2)$. From this point of view the problem of separation of variables in equation (5.3.1) is studied in [37, 38, 179, 257].

But of no less importance is the problem of describing the potentials $V(x_1,x_2)$ for which the Schrödinger equation admits variable separation. Thus by a separation of variables in equation (5.3.1) we imply two mutually connected problems. The first one is to describe all such functions $V(x_1,x_2)$ that the corresponding Schrödinger equation (5.3.1) can be separated into three ODEs in some coordinate system of the form (5.3.2) (classification problem). The second problem is to construct for each function $V(x_1,x_2)$ obtained in this way all coordinate systems (5.3.2) enabling us to carry out separation of variables in equation (5.3.1).

As far as we know, the second problem has been solved provided $V = 0$ [38] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$[37]. The first one was considered in a restricted sense in [257]. Using the symmetry approach to classification problem the authors obtained some potentials providing separability of equation (5.3.1) and carried out separation of variables in the corresponding Schrödinger equations. But their results are far from being complete and systematic. The necessary and sufficient conditions imposed on the potential $V(x_1,x_2)$ by the requirement that the Schrödinger equation admits symmetry operators of an arbitrary order are obtained in [231]. But so far there is no systematic and exhaustive description of potentials $V(x_1,x_2)$ providing separation of variables in equation (5.3.1).

To have a right to claim a description of *all* potentials and *all* coordinate systems, which make it possible to separate the Schrödinger equation, it is necessary to have a definition of separation of variables. It is natural to utilize Definition 5.1.3 adapted to the case of a second-order PDE with one dependent variable. Consider the following system of ODEs:

$$\begin{aligned}i\frac{d\varphi_0}{dt} &= U_0(t,\varphi_0;\lambda_1,\lambda_2),\\ \frac{d^2\varphi_1}{d\omega_1^2} &= U_1\left(\omega_1,\varphi_1,\frac{d\varphi_1}{d\omega_1};\lambda_1,\lambda_2\right),\end{aligned} \tag{5.3.3}$$

$$\frac{d^2\varphi_2}{d\omega_2^2} \quad = \quad U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right),$$

where $U_0,\ U_1,\ U_2$ are some smooth functions of the corresponding arguments, $\{\lambda_1, \lambda_2\} \subset \mathbb{R}^1$ are arbitrary parameters (separation constants) and what is more

$$\operatorname{rank} \|\partial U_\mu / \partial \lambda_a\|_{\mu=0}^{2}\,{}_{a=1}^{2} = 2 \tag{5.3.4}$$

(the last condition ensures essential dependence of the corresponding solution with separated variables on $\lambda_1,\ \lambda_2$, see [201]).

Definition 5.3.1. We say that equation (5.3.1) admits a separation of variables in the coordinate system $t,\ \omega_1(t, \vec{x}),\ \omega_2(t, \vec{x})$ if there exists such a function $Q(t, \vec{x})$ that substitution of the Ansatz

$$u = Q(t, \vec{x})\varphi_0(t)\varphi_1\Big(\omega_1(t, \vec{x})\Big)\varphi_2\Big(\omega_2(t, \vec{x})\Big) \tag{5.3.5}$$

into (5.3.1) with subsequent elimination of the derivatives $\dot{\varphi}_0,\ \ddot{\varphi}_1,\ \ddot{\varphi}_2$ according to equations (5.3.3) yields an identity with respect to $\varphi_0,\ \varphi_1,\ \varphi_2,\ \dot{\varphi}_1,\ \dot{\varphi}_2,\ \lambda_1,\ \lambda_2$. Thus, according to the above definition to separate variables in equation (5.3.1) we have

- to substitute the expression (5.3.5) into (5.3.1),
- to eliminate the derivatives $\dot{\varphi}_0,\ \ddot{\varphi}_1,\ \ddot{\varphi}_2$ with the help of equations (5.3.3),
- to split the equality obtained with respect to the variables $\varphi_0,\ \varphi_1,\ \varphi_2,\ \dot{\varphi}_1,\ \dot{\varphi}_2,\ \lambda_1,\ \lambda_2$ considered as independent.

As a result, we get some over-determined system of PDEs for the functions $Q(t, \vec{x}),\ \ \omega_1(t, \vec{x}),\ \ \omega_2(t, \vec{x})$. On solving it we obtain a complete description of all coordinate systems and potentials providing separation of variables in the Schrödinger equation. Naturally, the words *complete description* make sense only within the framework of our definition. So if one uses a more general definition it may be possible to construct new coordinate systems and potentials providing separability of equation (5.3.1). But all solutions of the Schrödinger equation with separated variables known to us fit into the scheme suggested by us and can be obtained in the above described way.

1. Classification of potentials $V(x_1, x_2)$. We do not adduce in full detail computations needed because they are very cumbersome. We will restrict ourselves to pointing out main steps of the realization of the above suggested algorithm.

First of all we make a remark, which makes life a little bit easier. It is readily seen that a substitution of the form

$$\begin{aligned} Q &\to Q' = Q\Psi_1(\omega_1)\Psi_2(\omega_2), \\ \omega_a &\to \omega'_a = \Omega_a(\omega_a), \quad a = 1, 2, \\ \lambda_a &\to \lambda'_a = \Lambda_a(\lambda_1, \lambda_2), \quad a = 1, 2, \end{aligned} \tag{5.3.6}$$

does not alter the structure of relations (5.3.3), (5.3.4), (5.3.5). That is why we can introduce the following equivalence relation:

$$(\omega_1,\ \omega_2,\ Q)\ \sim\ (\omega'_1,\ \omega'_2,\ Q'),$$

provided (5.3.6) holds with some $\Psi_a,\ \Omega_a,\ \Lambda_a$.

Substituting (5.3.5) into (5.3.1) and excluding the derivatives $\dot\varphi_0,\ \ddot\varphi_1,\ \ddot\varphi_2$ with the use of equations (5.3.3) we get

$$\begin{aligned} & i(Q_t\varphi_0\varphi_1\varphi_2 + QU_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) \\ & \quad + (\triangle Q)\varphi_0\varphi_1\varphi_2 + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 \\ & \quad + Q\Big((\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{1x_a}\omega_{1x_a}\varphi_0U_1\varphi_2 \\ & \quad + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1U_2 + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\Big) = VQ\varphi_0\varphi_1\varphi_2, \end{aligned}$$

where the summation over the repeated index a from 1 to 2 is understood, $\triangle = \partial^2_{x_1} + \partial^2_{x_2}$.

Splitting the equality obtained with respect to the independent variables $\varphi_1,\ \varphi_2,\ \dot\varphi_1,\ \dot\varphi_2,\ \ \lambda_1,\ \lambda_2$ we conclude that ODEs (5.3.3) are linear and up to the equivalence relation (5.3.6) can be written in the form

$$\begin{aligned} i\frac{d\varphi_0}{dt} &= \Big(\lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t)\Big)\varphi_0, \\ \frac{d^2\varphi_1}{d\omega_1^2} &= \Big(\lambda_1 B_{11}(\omega_1) + \lambda_2 B_{12}(\omega_1) + B_{01}(\omega_1)\Big)\varphi_1, \\ \frac{d^2\varphi_2}{d\omega_2^2} &= \Big(\lambda_1 B_{21}(\omega_2) + \lambda_2 B_{22}(\omega_2) + B_{02}(\omega_2)\Big)\varphi_2 \end{aligned}$$

and what is more, functions $\omega_1,\ \omega_2,\ Q$ satisfy an over-determined system of nonlinear PDEs

1) $\omega_{1x_b}\omega_{2x_b} = 0,$

2) $B_{1a}(\omega_1)\omega_{1x_b}\omega_{1x_b} + B_{2a}(\omega_2)\omega_{2x_b}\omega_{2x_b} + R_a(t) = 0,$

$$
\begin{aligned}
&3)\quad 2\omega_{ax_b}Q_{x_b} + Q(i\omega_{at} + \triangle\omega_a) = 0, \qquad (5.3.7)\\
&4)\quad \Big(B_{01}(\omega_1)\omega_{1x_b}\omega_{1x_b} + B_{02}(\omega_1)\omega_{2x_b}\omega_{2x_b}\Big)Q + iQ_t + \triangle Q\\
&\qquad +R_0(t)Q - V(x_1,x_2)Q = 0,
\end{aligned}
$$

where $a, b = 1, 2$.

Thus, to solve the problem of separation of variables for the linear Schrödinger equation it is necessary to construct a general solution of the system of nonlinear PDEs (5.3.7). Roughly speaking, to solve a linear equation we have to solve a system of *nonlinear equations*! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional d'Alembert equation [226].

However in the present case we have succeeded in integrating the system of nonlinear PDEs (5.3.7). Our approach to its integration is based on the following change of variables (hodograph transformation)

$$
\begin{aligned}
&z_0 = t,\ z_1 = Z_1(t,\omega_1,\omega_2),\ z_2 = Z_2(t,\omega_1,\omega_2),\\
&v_1 = x_1,\ v_2 = x_2,
\end{aligned}
$$

where $z_0,\ z_1,\ z_2$ are new independent and $v_1,\ v_2$ are new dependent variables, correspondingly.

Using the hodograph transformation determined above we have constructed the general solution of equations 1–3 from (5.3.7). It is given up to the equivalence relation (5.3.6) by one of the following formulae:

$$
\begin{aligned}
&1)\quad \omega_1 = A(t)x_1 + W_1(t), \quad \omega_2 = B(t)x_2 + W_2(t),\\
&\quad Q(t,\vec{x}) = \exp\Big\{-(i/4)\Big((\dot{A}/A)x_1^2 + (\dot{B}/B)x_2^2\Big) - (i/2)\Big((\dot{W}_1/A)x_1\\
&\qquad + (\dot{W}_2/B)x_2\Big)\Big\};\\
&2)\quad x_1 = W(t)\mathrm{e}^{\omega_1}\sin\omega_2 + W_1(t), \quad x_2 = W(t)\mathrm{e}^{\omega_1}\cos\omega_2 + W_2(t),\\
&\quad Q(t,\vec{x}) = \exp\Big\{(i\dot{W}/4W)\Big((x_1-W_1)^2 + (x_2-W_2)^2\Big)\\
&\qquad + (i/2)(\dot{W}_1x_1 + \dot{W}_2x_2)\Big\};\\
&3)\quad x_1 = (1/2)W(t)(\omega_1^2 - \omega_2^2) + W_1(t), \quad x_2 = W(t)\omega_1\omega_2 + W_2(t), \qquad (5.3.8)\\
&\quad Q(t,\vec{x}) = \exp\Big\{(i\dot{W}/4W)\Big((x_1-W_1)^2 + (x_2-W_2)^2\Big)\\
&\qquad + (i/2)(\dot{W}_1x_1 + \dot{W}_2x_2)\Big\};\\
&4)\quad x_1 = W(t)\cosh\omega_1\cos\omega_2 + W_1(t), \quad x_2 = W(t)\sinh\omega_1\sin\omega_2 + W_2(t),
\end{aligned}
$$

$$Q(t,\vec{x}) = \exp\Big\{(i\dot{W}/4W)\Big((x_1 - W_1)^2 + (x_2 - W_2)^2\Big) + (i/2)(\dot{W}_1 x_1 + \dot{W}_2 x_2)\Big\}.$$

Here $A,\ B,\ W,\ W_1,\ W_2$ are arbitrary smooth functions of t.

Substituting the obtained expressions for the functions $Q,\ \omega_1,\ \omega_2$ into the last equation from the system (5.3.7) and splitting with respect to variables $x_1,\ x_2$ we get explicit forms of potentials $V(x_1,x_2)$ and systems of nonlinear ODEs for unknown functions $A(t),\ B(t),\ W(t),\ W_1(t),\ W_2(t)$. We have succeeded in integrating these and in constructing all coordinate systems providing the separation of variables in the initial equation (5.3.1) [316]. Integration has been carried out up to the equivalence relation which is introduced below in Notes 5.3.1–5.3.3.

Note 5.3.1. The Schrödinger equation with the potential

$$V(x_1,x_2) = k_1x_1 + k_2x_2 + k_3 + V_1(k_2x_1 - k_1x_2), \tag{5.3.9}$$

where $k_1,\ k_2,\ k_3$ are constants, is transformed to the Schrödinger equation with the potential

$$V'(x_1',x_2') = V_1(k_2x_1' - k_1x_2') \tag{5.3.10}$$

by means of the following change of variables:

$$\begin{aligned} &t' = t, \quad \vec{x}' = \vec{x} + t^2\vec{k},\\ &u' = u\exp\{(i/3)(k_1^2 + k_2^2)t^3 + it(k_1x_1 + k_2x_2) + ik_3t\}. \end{aligned} \tag{5.3.11}$$

It is readily seen that the class of Ansätze (5.3.5) is transformed into itself by the above change of variables. That is why potentials (5.3.9) and (5.3.10) are considered as equivalent.

Note 5.3.2. The Schrödinger equation with the potential

$$V(x_1,x_2) = k(x_1^2 + x_2^2) + V_1(x_1/x_2)(x_1^2 + x_2^2)^{-1} \tag{5.3.12}$$

with $k = \mathrm{const}$ is reduced to the Schrödinger equation with the potential

$$V'(x_1,x_2) = V_1(x_1'/x_2')(x_1'^2 + x_1'^2)^{-1} \tag{5.3.13}$$

by means of the change of variables

$$t' = \alpha(t), \quad \vec{x}' = \beta(t)\vec{x}, \quad u' = u\exp\Big\{i\gamma(t)(x_1^2 + x_2^2) + \delta(t)\Big\},$$

where $\Big(\alpha(t),\ \beta(t),\ \gamma(t),\ \delta(t)\Big)$ is an arbitrary solution of the system of ODEs

$$\dot\gamma - 4\gamma^2 = k, \quad \dot\beta - 4\gamma\beta = 0, \quad \dot\alpha - \beta^2 = 0, \quad \dot\delta + 4\gamma = 0$$

such that $\beta \neq 0$.

Since the above change of variables does not alter the structure of the Ansatz (5.3.5), when classifying potentials $V(x_1, x_2)$ providing separability of the corresponding Schrödinger equation we consider potentials (5.3.12), (5.3.13) as equivalent.

Note 5.3.3. It is well-known (see e.g. [177, 232]) that the general form of the invariance group admitted by equation (5.3.1) is as follows:

$$\begin{aligned} &t' = F(t, \vec\theta), \quad x'_a = g_a(t, \vec x, \vec\theta), \quad a = 1, 2, \\ &u' = h(t, \vec x, \vec\theta)u + U(t, \vec x), \end{aligned}$$

where $\vec\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ are group parameters and $U(t, \vec x)$ is an arbitrary solution of equation (5.3.1).

The above transformations also do not alter the structure of the Ansatz (5.3.5). That is why systems of coordinates $t',\ x'_1,\ x'_2$ and $t,\ x_1,\ x_2$ are considered as equivalent.

Below we give without derivation a list of potentials $V(x_1, x_2)$ providing separability of the Schrödinger equation (5.3.1) (some details can be found in [316]).

1. $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$;
   (a) $V(x_1, x_2) = k_1x_1^2 + k_2x_1^{-2} + V_2(x_2), \quad k_2 \neq 0$;
      i. $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_1^{-2} + k_4x_2^{-2}, \quad k_3k_4 \neq 0,$ $k_1^2 + k_2^2 \neq 0,\ k_1 \neq k_2$;
      ii. $V(x_1, x_2) = k_1x_1^2 + k_2x_1^{-2}, \quad k_1k_2 \neq 0$;
      iii. $V(x_1, x_2) = k_1x_1^{-2} + k_2x_2^{-2}$;

   (b) $V(x_1, x_2) = k_1x_1^2 + V_2(x_2)$;
      i. $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}, \quad k_1k_3 \neq 0,\ k_1 \neq k_2$;
      ii. $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2, \quad k_1k_2 \neq 0,\ k_1 \neq k_2$;
      iii. $V(x_1, x_2) = k_1x_1^2 + k_2x_2^{-2}, \quad k_1 \neq 0$;

2. $V(x_1, x_2) = V_1(x_1^2 + x_2^2) + V_2\left(\dfrac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}$;

(a) $V(x_1, x_2) = V_2\left(\dfrac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}$;

(b) $V(x_1, x_2) = (x_1^2 + x_2^2)^{-1/2}(k_1 + k_2 x_1 x_2^{-2}) + k_3 x_2^{-2}, \quad k_1^2 + k_3^2 \neq 0$;

3. $V(x_1, x_2) = \Big(V_1(\omega_1) + V_2(\omega_2)\Big)(\omega_1^2 + \omega_2^2)^{-1}$,

   where $\quad \omega_1^2 - \omega_2^2 = 2x_1,\ \omega_1\omega_2 = x_2$;

4. $V(x_1, x_2) = \Big(V_1(\omega_1) + V_2(\omega_2)\Big)(\sinh^2\omega_1 + \sin^2\omega_2)^{-1}$,

   where $\quad \cosh\omega_1 \cos\omega_2 = x_1,\ \sinh\omega_1 \sin\omega_2 = x_2$;

5. $V(x_1, x_2) = 0$.

In the above formulae $V_1,\ V_2$ are arbitrary smooth functions, $k_1,\ k_2,\ k_3,\ k_4$ are real arbitrary constants.

It should be emphasized that the above potentials are not inequivalent in a usual sense. These potentials differ from each other by the fact that the coordinate systems providing separability of the corresponding Schrödinger equations are different. Moreover, in some cases the form of coordinate systems depends essentially on the signs of the parameters $k_j,\ j = 1, \ldots, 4$.

Next, we consider in detail separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator potential $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2$ and the Coulomb potential $V(x_1, x_2) = k_1(x_1^2 + x_2^2)^{-1/2}$.

2. Separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator and the Coulomb potentials. Here we will obtain all coordinate systems providing separability of the Schrödinger equation with the potential $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2$

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = (k_1 x_1^2 + k_2 x_2^2)u. \tag{5.3.14}$$

In the following, we consider the case $k_1 \neq k_2$, because otherwise equation (5.3.1) is reduced to the free Schrödinger equation (see Note 5.3.2) which has been studied in detail in [226].

Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters $k_1,\ k_2$. We consider in some detail the case, when $k_1 < 0,\ k_2 > 0$ (the cases $k_1 > 0,\ k_2 > 0$ and $k_1 < 0,\ k_2 < 0$ are handled in an analogous way). This means that equation (5.3.14) can be written in the form

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + (1/4)(a^2 x_1^2 - b^2 x_2^2)u = 0. \tag{5.3.15}$$

where $a,\ b$ are arbitrary non-zero real constants (the factor 1/4 is introduced for further convenience).

As stated above to describe all coordinate systems $t,\ \omega_1(t,\vec{x}),\ \omega_2(t,\vec{x})$ providing separability of equation (5.3.14) it is necessary to construct the general solution of system (5.3.8) with $V(x_1,x_2) = -(1/4)(a^2x_1^2 - b^2x_2^2)$. The general solution of equations 1–3 from (5.3.7) splits into four inequivalent classes listed in (5.3.8). Analysis shows that only solutions belonging to the first class can satisfy the fourth equation of (5.3.7).

Substituting the expressions for $\omega_1,\ \omega_2,\ Q$ given by the formulae 1 from (5.3.8) into the equation 4 from (5.3.7) with $V(x_1,x_2) = -(1/4)(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to $x_1,\ x_2$ yield

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$(\dot{A}/A)^{\cdot} - (\dot{A}/A)^2 - 4\alpha_1A^4 + a^2 = 0, \tag{5.3.16}$$

$$(\dot{B}/B)^{\cdot} - (\dot{B}/B)^2 - 4\beta_1B^4 - b^2 = 0, \tag{5.3.17}$$

$$\ddot{\theta}_1 - 2\dot{\theta}_1(\dot{A}/A) - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{5.3.18}$$

$$\ddot{\theta}_2 - 2\dot{\theta}_2(\dot{B}/B) - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0. \tag{5.3.19}$$

Here $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ are arbitrary real constants.

Evidently, equations (5.3.16)–(5.3.19) can be rewritten in the following unified form:

$$(\dot{y}/y)^{\cdot} - (\dot{y}/y)^2 - 4\alpha y^4 = k, \quad \ddot{z} - 2\dot{z}(\dot{y}/y) - 2(2\alpha z + \beta)y^4 = 0. \tag{5.3.20}$$

Provided $k = -a^2 < 0$, system (5.3.20) coincides with equations (5.3.16), (5.3.18) and under $k = b^2 > 0$ with equations (5.3.17), (5.3.19).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Indeed, the coordinate functions ω_a have the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system $t,\ \omega_1,\ \omega_2$ is equivalent to the coordinate system $t,\ \omega_1 + C_1,\ \omega_2 + C_2,\ C_a \in \mathbb{R}^1$. Hence it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we can choose in (5.3.20) $\beta = 0$.

Case 1. $\alpha = 0$

On making in (5.3.20) the change of variables

$$w = \dot{y}/y, \quad v = z/y \tag{5.3.21}$$

we get

$$\dot{w} = w^2 + k, \quad \ddot{v} + kv = 2\beta y^3. \tag{5.3.22}$$

First, we consider the case $k = -a^2 < 0$. Then, the general solution of the first equation from (5.3.22) is given by one of the formulae

$$w = -a \coth a(t + C_1), \quad w = -a \tanh a(t + C_1), \quad w = \pm a, \quad C_1 \in \mathbb{R}^1,$$

whence

$$\begin{aligned} &y(t) = C_2 \sinh^{-1} a(t + C_1), \quad y(t) = C_2 \cosh^{-1} a(t + C_1),\\ &y(t) = C_2 \exp(\pm at), \quad C_2 \in \mathbb{R}^1. \end{aligned} \tag{5.3.23}$$

The second equation of system (5.3.22) is a linear inhomogeneous ODE. We substitute its general solution into (5.3.21) and get the following expressions for $z(t)$:

$$\begin{aligned} z(t) &= (C_3 \cosh at + C_4 \sinh at) \sinh^{-1} a(t + C_1)\\ &\quad +(\beta C_2^4/a^2) \sinh^{-2} a(t + C_1),\\ z(t) &= (C_3 \cosh at + C_4 \sinh at) \cosh^{-1} a(t + C_1)\\ &\quad +(\beta C_2^4/a^2) \cosh^{-2} a(t + C_1),\\ z(t) &= (C_3 \cosh at + C_4 \sinh at) \exp(\pm at)\\ &\quad +(\beta C_2^4/4a^2) \exp(\pm 4at), \end{aligned} \tag{5.3.24}$$

where $\{C_3, C_4\} \subset \mathbb{R}^1$.

The case $k = b^2 > 0$ is treated in a similar way, the general solution of (5.3.20) being given by the formulae

$$\begin{aligned} y(t) &= D_2 \sin^{-1} b(t + D_1),\\ z(t) &= (D_3 \cos bt + D_4 \sin bt) \sin^{-1} b(t + D_1)\\ &\quad +(\beta D_2^4/b^2) \sin^{-2} b(t + D_1), \end{aligned} \tag{5.3.25}$$

where $D_1,\ D_2,\ D_3,\ D_4$ are arbitrary real constants.

Case 2. $\alpha \neq 0,\ \beta = 0$

On making in (5.3.20) the change of variables

$$y = \exp w, \quad v = z/y$$

we have

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{5.3.26}$$

The first ODE from (5.3.26) is reduced to the first-order linear ODE

$$(1/2)\frac{dp(w)}{dw} - p(w) = k + \alpha \exp 4w$$

by the substitution $\dot{w} = [p(w)]^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

The equation $\dot{w} = [p(w)]^{1/2}$ has a singular solution $w = C =$ const such that $p(C) = 0$. If $\dot{w} \neq 0$, then integrating the equation $\dot{w} = p(w)$ and returning to the initial variable y we get

$$\int^{y(t)} \tau^{-1}(\alpha\tau^4 + \gamma\tau^2 - k)^{-1/2} d\tau = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from (5.3.20). It is given by the following formulae:

under $k = -a^2 < 0$

$$\begin{aligned} y(t) &= C_2\Big(\alpha + \sinh 2a(t + C_1)\Big)^{-1/2}, \\ y(t) &= C_2\Big(\alpha + \cosh 2a(t + C_1)\Big)^{-1/2}, \\ y(t) &= C_2\Big(\alpha + \exp(\pm 2at)\Big)^{-1/2}, \end{aligned} \tag{5.3.27}$$

under $k = b^2 > 0$

$$y(t) = D_2\Big(\alpha + \sin 2b(t + D_1)\Big)^{-1/2}. \tag{5.3.28}$$

Here $C_1,\ C_2,\ D_1,\ D_2$ are arbitrary real constants.

Integrating the second ODE from (5.3.26) and returning to the initial variable z we have

under $k = -a^2 < 0$

$$z(t) = y(t)(C_3 \cosh at + C_4 \sinh at), \tag{5.3.29}$$

under $k = b^2 > 0$

$$z(t) = y(t)(D_3 \cos bt + D_4 \sin bt), \tag{5.3.30}$$

where $C_3,\ C_4,\ D_3,\ D_4$ are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODEs (5.3.20) which is given by the formulae (5.3.24)–(5.3.30).

Substitution of the formulae (5.3.21), (5.3.23)–(5.3.25), (5.3.27)–(5.3.30) into the corresponding expressions 1 from (5.3.8) yields a complete list of coordinate systems providing separability of the Schrödinger equation (5.3.15). These systems can be transformed to canonical form if we use Note 5.3.3.

The invariance group of equation (5.3.15) is generated by the following basis operators [315]:

$$\begin{aligned}
&P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad Q_\infty = U(t,\vec{x})\partial_u,\\
&P_1 = \cosh at\, \partial_{x_1} + (ia/2)(x_1 \sinh at)u\partial_u,\\
&P_2 = \cos bt\, \partial_{x_2} - (ib/2)(x_2 \sin bt)u\partial_u,\\
&G_1 = \sinh at\, \partial_{x_1} + (ia/2)(x_1 \cosh at)u\partial_u,\\
&G_2 = \sin bt\, \partial_{x_2} + (ib/2)(x_2 \cos bt)u\partial_u,
\end{aligned} \tag{5.3.31}$$

where $U(t,\vec{x})$ is an arbitrary solution of equation (5.3.15).

Making use of the finite transformations generated by the infinitesimal operators (5.3.31) and Note 5.3.3 we can choose in the formulae (5.3.23)–(5.3.25), (5.3.27), (5.3.29), (5.3.30) $C_3 = C_4 = D_1 = 0,\ D_3 = D_4 = 0,\ C_2 = D_2 = 1$. As a result, we come to the following assertion.

Theorem 5.3.1. *The Schrödinger equation (5.3.15) admits separation of variables in 21 inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t,\vec{x}), \quad \omega_2 = \omega_2(t,\vec{x}), \tag{5.3.32}$$

where ω_1 is given by one of the formulae from the first and ω_2 by one of the formulae from the second column of the Table 5.3.1.

There is no necessity to consider specially the case when in (5.3.14) $k_1 > 0,\ k_2 < 0$, since such an equation by the change of independent variables $u(t,x_1,x_2) \to u(t,x_2,x_1)$ is reduced to equation (5.3.15).

Below we adduce without proof the assertions describing coordinate systems providing separation of variables in equation (5.3.14) with $k_1 < 0,\ k_2 < 0$ and $k_1 > 0,\ k_2 > 0$ and in the Schrödinger equation with the Coulomb potential $k_1(x_1^2 + x_2^2)^{-1/2}$.

Table 5.3.1. **Coordinate systems providing separability of the Schrödinger equation (5.3.15)**

$\omega_1(t,\vec{x})$	$\omega_2(t,\vec{x})$
$x_1\Big(\sinh a(t+C)\Big)^{-1}+\alpha\Big(\sinh a(t+C)\Big)^{-2}$	$x_2(\sin bt)^{-1}+\beta(\sin bt)^{-2}$
$x_1\Big(\cosh a(t+C)\Big)^{-1}+\alpha\Big(\cosh a(t+C)\Big)^{-2}$	$x_2(\beta+\sin 2bt)^{-1/2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	x_2
$x_1\Big(\alpha+\sinh 2a(t+C)\Big)^{-1/2}$	
$x_1\Big(\alpha+\cosh 2a(t+C)\Big)^{-1/2}$	
$x_1\Big(\alpha+\exp(\pm 2at)\Big)^{-1/2}$	
x_1	

Here $C,\ \alpha,\ \beta$ are arbitrary real constants.

Theorem 5.3.2. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}+(1/4)(a^2x_1^2+b^2x_2^2)u=0 \qquad (5.3.33)$$

with $a^2\neq 4b^2$ admits separation of variables in 49 inequivalent coordinate systems of the form (5.3.32), where ω_1 is given by one of the formulae from the first and ω_2 by one of the formulae from the second column of the Table 5.3.2. Provided $a^2=4b^2$, one more coordinate system should be included into the above list, namely,

$$\omega_0=t,\quad \omega_1^2-\omega_2^2=2x_1,\quad \omega_1\omega_2=x_2. \qquad (5.3.34)$$

Theorem 5.3.3. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-(1/4)(a^2x_1^2+b^2x_2^2)u=0 \qquad (5.3.35)$$

with $a^2\neq 4b^2$ admits separation of variables in 9 inequivalent coordinate systems of the form (5.3.32), where ω_1 is given by one of the formulae from the first and ω_2 by one of the formulae from the second column of the Table 5.3.3. Provided $a^2=4b^2$, the above list should be supplemented by the coordinate system (5.3.34).

Table 5.3.2. **Coordinate systems providing separability of the Schrödinger equation (5.3.33)**

$\omega_1(t,\vec{x})$	$\omega_2(t,\vec{x})$
$x_1\Big(\sinh a(t+C)\Big)^{-1}+\alpha\Big(\sinh a(t+C)\Big)^{-2}$	$x_2(\sinh bt)^{-1}+\beta(\sinh bt)^{-2}$
$x_1\Big(\cosh a(t+C)\Big)^{-1}+\alpha\Big(\cosh a(t+C)\Big)^{-2}$	$x_2(\cosh bt)^{-1}+\beta(\cosh bt)^{-2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	$x_2\exp(\pm bt)+\beta\exp(\pm 4bt)$
$x_1\Big(\alpha+\sinh 2a(t+C)\Big)^{-1/2}$	$x_2(\beta+\sinh 2bt)^{-1/2}$
$x_1\Big(\alpha+\cosh 2a(t+C)\Big)^{-1/2}$	$x_2(\beta+\cosh 2bt)^{-1/2}$
$x_1\Big(\alpha+\exp(\pm 2at)\Big)^{-1/2}$	$x_2\Big(\beta+\exp(\pm 2bt)\Big)^{-1/2}$
x_1	x_2

Here $C,\ \alpha,\ \beta$ are arbitrary constants.

Theorem 5.3.4. *The Schrödinger equation with the Coulomb potential*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-k_1(x_1^2+x_2^2)^{-1/2}u=0$$

admits separation of variables in two coordinate systems. One of them is the polar coordinate system

$$t=\omega_0,\quad x_1=\mathrm{e}^{\omega_1}\sin\omega_2,\quad x_2=\mathrm{e}^{\omega_1}\cos\omega_2$$

and another is the parabolic coordinate system (5.3.34).

It is important to note that explicit forms of coordinate systems providing separability of equations (5.3.15), (5.3.33), (5.3.35) depend essentially on the parameters $a,\ b$ contained in the potential $V(x_1,x_2)$. It means that the free Schrödinger equation ($V=0$) does not admit separation of variables in such coordinate systems. Consequently, they are essentially new.

3. Conclusion. In the present section we have studied the case when the Schrödinger equation (5.3.1) separates into one first-order and two second-order ODEs. It is not difficult to prove that there are no functions $Q(t,\vec{x})$, $\omega_\mu(t,\vec{x}),\ \mu=0,\dots,2$ such that the Ansatz

$$u=Q(t,\vec{x})\varphi_0\Big(\omega_0(t,\vec{x})\Big)\varphi_1\Big(\omega_1(t,\vec{x})\Big)\varphi_2\Big(\omega_2(t,\vec{x})\Big)$$

separates equation (5.3.1) into three second-order ODEs (see [315]). Nevertheless, there exists a possibility for equation (5.3.1) to be separated into two first-order and one second-order ODEs or into three first-order ODEs. This is a probable source of new potentials and new coordinate systems providing separability of the Schrödinger equation. It should be mentioned that separation of the two-dimensional d'Alembert equation

$$u_{tt} - u_{xx} = V(x)u$$

into one first-order and one second-order ODEs gives no new potentials as compared with separation of it into two second-order ODEs. But for some already known potentials new coordinate systems providing separability of the above equation are obtained [312, 314].

Table 5.3.3. **Coordinate systems providing separability of the Schrödinger equation (5.3.35)**

$\omega_1(t,\vec{x})$	$\omega_2(t,\vec{x})$
$x_1\Big(\sin a(t+C)\Big)^{-1} + \alpha\Big(\sin a(t+C)\Big)^{-2}$	$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}$
$x_1\Big(\beta + \sin 2a(t+C)\Big)^{-1/2}$	$x_2(\beta + \sin 2bt)^{-1/2}$
x_1	x_2

Here $C,\ \alpha,\ \beta$ are arbitrary constants.

Let us briefly analyze the connection between separability and symmetry properties of equation (5.3.1). It is well-known that each solution of the free Schrödinger equation with separated variables is a common eigenfunction of its two mutually commuting second-order symmetry operators [226]. And what is more, separation constants $\lambda_1,\ \lambda_2$ are eigenvalues of these symmetry operators.

We will establish that the same assertion holds for the Schrödinger equation (5.3.1). Let us make in equation (5.3.1) the following change of variables:

$$u = Q(t,\vec{x})U\Big(t, \omega_1(t,\vec{x}), \omega_2(t,\vec{x})\Big), \tag{5.3.36}$$

where $(Q,\ \omega_1,\ \omega_2)$ is an arbitrary solution of the system of PDEs (5.3.7).

Substituting the expression (5.3.36) into (5.3.1) and taking into account equations (5.3.7) we get

$$\begin{gathered}Q\Big(iU_t + [U_{\omega_1\omega_1} - B_{01}(\omega_1)U]\omega_{1x_a}\omega_{1x_a} + [U_{\omega_2\omega_2} - B_{02}(\omega_2)U] \\ \times\omega_{2x_a}\omega_{2x_a}\Big) = 0.\end{gathered} \tag{5.3.37}$$

Resolving equations 2 from the system (5.3.7) with respect to $\omega_{1x_a}\omega_{1x_a}$ and $\omega_{2x_a}\omega_{2x_a}$ we have

$$\begin{aligned}\omega_{1x_a}\omega_{1x_a} &= (1/\delta)\Big(R_2(t)B_{21}(\omega_2) - R_1(t)B_{22}(\omega_2)\Big), \\ \omega_{2x_a}\omega_{2x_a} &= (1/\delta)\Big(R_1(t)B_{12}(\omega_1) - R_2(t)B_{11}(\omega_1)\Big),\end{aligned}$$

where $\delta = B_{11}(\omega_1)B_{22}(\omega_2) - B_{12}(\omega_1)B_{21}(\omega_2)$ ($\delta \neq 0$ resulting from the condition (5.3.4)).

Substitution of the above equalities into equation (5.3.37) with subsequent division by $Q \neq 0$ yields the following PDE:

$$\begin{gathered}iU_t + (1/\delta)R_1(t)\Big(B_{12}(\omega_1)[U_{\omega_2\omega_2} - B_{02}(\omega_2)U] - B_{22}(\omega_2) \\ \times[U_{\omega_1\omega_1} - B_{01}(\omega_1)U]\Big) + (1/\delta)R_2(t)\Big(B_{21}(\omega_2)[U_{\omega_1\omega_1} \\ -B_{01}(\omega_1)U] - B_{11}(\omega_1)[U_{\omega_2\omega_2} - B_{02}(\omega_2)U]\Big) = 0.\end{gathered} \tag{5.3.38}$$

Thus, in the new coordinates $t,\ \omega_1,\ \omega_2,\ U(t,\omega_1,\omega_2)$ equation (5.3.1) takes the form (5.3.38).

By direct (and very cumbersome) computation one can check that the following second-order differential operators

$$\begin{aligned}X_1 &= (1/\delta)B_{22}(\omega_2)\Big(\partial^2_{\omega_1} - B_{01}(\omega_1)\Big) - (1/\delta)B_{12}(\omega_1)\Big(\partial^2_{\omega_2} - B_{02}(\omega_2)\Big), \\ X_2 &= -(1/\delta)B_{21}(\omega_2)\Big(\partial^2_{\omega_1} - B_{01}(\omega_1)\Big) + (1/\delta)B_{11}(\omega_1)\Big(\partial^2_{\omega_2} - B_{02}(\omega_2)\Big)\end{aligned}$$

commute under arbitrary $B_{0a},\ B_{ab},\ a,b = 1,2$, i.e.,

$$[X_1,\ X_2] \equiv X_1X_2 - X_2X_1 = 0. \tag{5.3.39}$$

After being rewritten in terms of the operators $X_1,\ X_2$ equation (5.3.38) reads

$$\Big(i\partial_t - R_1(t)X_1 - R_2(t)X_2\Big)U = 0.$$

Since the relations

$$[i\partial_t - R_1(t)X_1 - R_2(t)X_2,\, X_a] = 0, \quad a = 1, 2 \tag{5.3.40}$$

hold, operators $X_1,\ X_2$ are mutually commuting symmetry operators of equation (5.3.38). Furthermore, the solution of equation (5.3.38) with separated variables $U = \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$ satisfies the identities

$$X_a U = \lambda_a U, \quad a = 1, 2. \tag{5.3.41}$$

Consequently, if we designate by $X_1',\ X_2'$ the operators $X_1,\ X_2$ written in the initial variables $t,\ \vec{x},\ u$, then we get from (5.3.39)–(5.3.41) the following equalities:

$$[i\partial_t + \triangle - V(x_1, x_2),\, X_a'] = 0, \quad a = 1, 2,$$

$$[X_1',\, X_2'] = 0, \qquad X_a' u = \lambda_a u, \quad a = 1, 2,$$

where $u = Q(t, \vec{x})\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$.

This means that each solution with separated variables is a common eigenfunction of two mutually commuting symmetry operators $X_1',\ X_2'$ of the Schrödinger equation (5.3.1), separation constants $\lambda_1,\ \lambda_2$ being their eigenvalues.

So, we have exposed two possible approaches to variable separation in linear PDEs which are based on their symmetry properties. The first one is to start with a set of commuting symmetry operators of the equation under study and to finish with the Ansatz (5.1.6) [12, 226, 255]. Another approach suggested for the first time in [169] is closer to the original understanding of the separation of variables in PDEs. A desired form (5.3.5) of the Ansatz for a solution with separated variables is postulated and then it turns out that the solution obtained can be related to a set of mutually-commuting symmetry operators of the equation under consideration.

Both approaches have their merits and drawbacks. We think that the utilization of the first approach is the only way to separate variables in multicomponent systems of PDEs. But to separate variables in PDEs with one dependent variable it is preferable to apply the second approach, since a computation of symmetry operators is an extra step which is not, in fact, necessary for obtaining solutions with separated variables. Another benefit of the approach in question is its simplicity, only some basics of the standard university course of mathematical physics are required for understanding and implementing it.

One more merit is that the second approach in contrast to the first one can be easily generalized in order to separate variables in *nonlinear* PDEs [314]. Using such a generalization we have classified in the paper [300] all nonlinear d'Alembert equations

$$u_{x_0x_0} - u_{x_1x_1} = F(u),$$

which separate into two first-order ODEs

$$\dot{\varphi}_1 = R_1(\varphi_1), \quad \dot{\varphi}_2 = R_1(\varphi_2)$$

by means of the Ansatz

$$u(x_0, x_1) = f\Big(\varphi_1(x_0) + \varphi_2(x_1)\Big).$$

It turned out that nonlinear d'Alembert equations admitting variable separation in the above sense are equivalent to one of the following PDEs:

$$\begin{aligned}
\Box u &= \lambda_1\Big(\cosh u + (\sinh 2u)\arctan \mathrm{e}^u\Big) + \lambda_2 \sinh 2u,\\
\Box u &= \lambda_1 \mathrm{e}^u + \lambda_2 \mathrm{e}^{-2u},\\
\Box u &= \lambda_1\Big(\sinh u - (\sinh 2u)\,\mathrm{arctanh}\, \mathrm{e}^u\Big) + \lambda_2 \sinh 2u,\\
\Box u &= \lambda_1\Big(2\sin u + (\sin 2u)\ln\tan(u/2)\Big) + \lambda_2 \sin 2u,\\
\Box u &= \lambda_1 u + \lambda_2 u \ln u,
\end{aligned}$$

where $\Box u = u_{x_0x_0} - u_{x_1x_1}$, λ_1, λ_2 are arbitrary real constants.

This fact enabled us to construct exact solutions of the above nonlinear PDEs which could not be found by the symmetry reduction procedure.

Let us also mention *anti-reduction* of PDEs [161, 162] which is also a generalization of a traditional notion of separation of variables specially designed to handle nonlinear PDEs.

CONDITIONAL SYMMETRY AND REDUCTION OF SPINOR EQUATIONS

In this chapter a non-Lie method of reduction of nonlinear Poincaré- and Galilei-invariant systems of PDEs to differential equations of lower dimension is suggested. With the use of this method we construct the wide classes of conditionally-invariant Ansätze reducing nonlinear $P(1,3)$- and $G(1,3)$-invariant spinor equations to systems of ODEs.

6.1. Non-Lie reduction of Poincaré-invariant spinor equations

In Section 2.3 we have constructed a number of Ansätze for the spinor field $\psi(x)$ reducing $P(1,3)$-invariant equation

$$\Big(i\gamma_\mu\partial_\mu - \tilde{f}_1(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi) - \tilde{f}_2(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi)\gamma_4\Big)\psi = 0 \tag{6.1.1}$$

to systems of ODEs which cannot be obtained within the framework of the classical Lie approach. Existence of such Ansätze is a consequence of *conditional symmetry* of equation (6.1.1).

Definition 6.1.1. Equation (6.1.1) is conditionally-invariant under the involutive set of operators

$$Q_a = \xi_{a\mu}(x)\partial_\mu + \eta_a(x), \quad a = 1, \dots, N,$$

if the system of PDEs

$$(i\gamma_\mu\partial_\mu - \tilde{f}_1 - \tilde{f}_2\gamma_4)\psi = 0, \quad Q_a\psi = 0, \quad a = 1, \dots, N \tag{6.1.2}$$

is invariant in Lie sense with respect to the one-parameter groups generated by the operators Q_a.

Due to Theorem 1.5.1 conditional invariance of PDE (6.1.1) under the involutive set of operators Q_a ensures its reducibility and, consequently, can be used to construct exact solutions of the (6.1.1).

A usual approach to investigation of conditional symmetry of a given PDE is application of the infinitesimal Lie method. But the problem is that the determining equations for functions $\xi_{a\mu}(x),\ \eta_a(x)$ prove to be nonlinear ones. That is why there is a little hope to describe all conditional symmetries of multi-dimensional system of PDEs (6.1.1). It should be said that more or less systematic results on conditional symmetry of PDEs are obtained for two-dimensional equations only [137].

In the present section we suggest a method making it possible to get both invariant and conditionally-invariant Ansätze constructed in Sections 2.2, 2.4. Moreover, applying this method we obtain some essentially new Ansätze for spinor field $\psi = \psi(x)$ reducing system of PDEs (6.1.1) to systems of ODEs.

1. Reduction of the nonlinear Dirac equation (6.1.1). Analysis of Ansätze for the spinor field invariant under the one- and three-parameter subgroups of the Poincaré group shows that all of them have the following structure:

$$\begin{aligned}\psi(x) \;&=\; \exp\{\theta_A\gamma_A(\gamma_0+\gamma_3)\}\exp\{(1/2)\theta_0\gamma_0\gamma_3+(1/2)\theta_3\gamma_1\gamma_2\}\\ &\quad\times\begin{cases}\varphi(\omega_1,\omega_2,\omega_3),\\ \varphi(\omega_1),\end{cases}\end{aligned} \tag{6.1.3}$$

where φ is an arbitrary four-component function-column; $\theta_\mu,\ \omega_a$ are some real-valued scalar functions, the constraint holding

$$\theta_A = \theta_A(x_0+x_3, x_1, x_2). \tag{6.1.4}$$

Hereafter the subscripts denoted by Latin alphabet letters A, B take the values 1, 2 and summation over the repeated indices is understood.

The key idea of the approach suggested can be formulated in a rather simple and natural way: we impose no *a priori* constraints on the functions θ_μ, ω_a, they are obtained from the requirement that substitution of expression (6.1.3) into (6.1.1) yields a system of PDEs for the function $\varphi(\vec{\omega})$ (or a system of ODEs for the function $\varphi(\omega_1)$) with coefficients depending on the new variables $\omega_1,\ \omega_2,\ \omega_3$ only.

In the following we describe all Ansätze of the form (6.1.3), (6.1.4) reducing the system of nonlinear four-dimensional PDEs (6.1.1) to a system of ODEs.

Substituting the Ansatz

$$\psi(x) = \exp\{\theta_A\gamma_A(\gamma_0+\gamma_3)\}\exp\{(1/2)\theta_0\gamma_0\gamma_3 + (1/2)\theta_3\gamma_1\gamma_2\}\varphi(\omega) \qquad (6.1.5)$$

into equation (6.1.1) and multiplying the expression obtained by the matrix

$$\exp\{-(1/2)\theta_0\gamma_0\gamma_3 - (1/2)\theta_3\gamma_1\gamma_2\}\exp\{-\theta_A\gamma_A(\gamma_0+\gamma_3)\}$$

on the left yield

$$iR_1\varphi + iR_2\dot\varphi = \Big(\tilde f_1(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi) + \tilde f_2(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\gamma_4\Big)\varphi, \qquad (6.1.6)$$

where $R_1 = R_1(x)$, $R_2 = R_2(x)$ are (4×4)-matrices determined by the following equalities:

$$\begin{aligned}
R_1 \;=\;& 2e^{\theta_0}\Big(-\partial_A\theta_A + \gamma_1\gamma_2(\partial_1\theta_2 - \partial_2\theta_1)\Big)(\gamma_0+\gamma_3) + \Big[(\gamma_0\partial_0\theta_0 + \gamma_3\partial_3\theta_0)\\
&\times(\cosh\theta_0 + \gamma_0\gamma_3\sinh\theta_0) + \gamma_A\Big(\partial_A\theta_0 + 2\theta_A(\partial_3\theta_0 - \partial_0\theta_0)\Big)(\cos\theta_3\\
&+\gamma_1\gamma_2\sin\theta_3) - 2e^{\theta_0}\theta_A(\partial_A\theta_0)(\gamma_0+\gamma_3)\Big]\gamma_0\gamma_3 + 2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_0\\
&-\partial_3\theta_0)(\gamma_0+\gamma_3) + \Big[(\gamma_0\partial_0\theta_3 + \gamma_3\partial_3\theta_3)(\cosh\theta_0 + \gamma_0\gamma_3\sinh\theta_0)\\
&+\gamma_A\Big(\partial_A\theta_3 + 2\theta_A(\partial_3\theta_3 - \partial_0\theta_3)\Big)(\cos\theta_3 + \gamma_1\gamma_2\sin\theta_3) - 2e^{\theta_0}\theta_A\\
&\times(\partial_A\theta_3)(\gamma_0+\gamma_3)\Big]\gamma_1\gamma_2 + 2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_3 - \partial_3\theta_3)(\gamma_0+\gamma_3)\gamma_1\gamma_2,\\
R_2 \;=\;& (\gamma_0\partial_0\omega + \gamma_3\partial_3\omega)(\cosh\theta_0 + \gamma_0\gamma_3\sinh\theta_0) + \gamma_A\Big(\partial_A\omega + 2\theta_A(\partial_3\omega\\
&-\partial_0\omega)\Big)(\cos\theta_3 + \gamma_1\gamma_2\sin\theta_3) - 2e^{\theta_0}\theta_A(\partial_A\omega)(\gamma_0+\gamma_3)\\
&+2e^{\theta_0}\theta_A\theta_A(\partial_0\omega - \partial_3\omega)(\gamma_0+\gamma_3).
\end{aligned}$$

Consequently, Ansatz (6.1.5) reduces equation (6.1.1) to a system of ODEs iff there exist such (4×4)-matrices $Q_1(\omega)$, $Q_2(\omega)$ that

$$R_1(x) = Q_1(\omega), \quad R_2(x) = Q_2(\omega). \qquad (6.1.7)$$

Expanding matrices $Q_1(\omega)$, $Q_2(\omega)$ in the complete system of the Dirac matrices and equating coefficients of the matrices I, γ_μ, $S_{\mu\nu}$, $\gamma_4\gamma_\mu$, γ_4 we obtain from (6.1.7) the over-determined system of nonlinear PDEs for functions θ_μ, ω

$$1)\quad (\partial_0\theta_0)\sinh\theta_0 + (\partial_3\theta_0)\cosh\theta_0 - 2e^{\theta_0}\partial_A\theta_A - 2e^{\theta_0}\theta_A\partial_A\theta_0$$

$$
\begin{aligned}
&\quad +2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_0-\partial_3\theta_0)=f_1(\omega),\\
&2)\quad (\partial_0\theta_0)\cosh\theta_0+(\partial_3\theta_0)\sinh\theta_0-2e^{\theta_0}\partial_A\theta_A-2e^{\theta_0}\theta_A\partial_A\theta_0\\
&\quad +2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_0-\partial_3\theta_0)=f_2(\omega),\\
&3)\quad \Big(\partial_2\theta_3+2\theta_2(\partial_3\theta_3-\partial_0\theta_3)\Big)\cos\theta_3-\Big(\partial_1\theta_3+2\theta_1(\partial_3\theta_3\\
&\quad -\partial_0\theta_3)\Big)\sin\theta_3=f_3(\omega),\\
&4)\quad \Big(\partial_1\theta_3+2\theta_1(\partial_3\theta_3-\partial_0\theta_3)\Big)\cos\theta_3+\Big(\partial_2\theta_3+2\theta_2(\partial_3\theta_3\\
&\quad -\partial_0\theta_3)\Big)\sin\theta_3=f_4(\omega),\\
&5)\quad 2e^{\theta_0}(\partial_1\theta_2-\partial_2\theta_1)+(\partial_0\theta_3)\cosh\theta_0+(\partial_3\theta_3)\sinh\theta_0\\
&\quad +2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_3-\partial_3\theta_3)-2e^{\theta_0}\theta_A\partial_A\theta_3=f_5(\omega),\\
&6)\quad 2e^{\theta_0}(\partial_1\theta_2-\partial_2\theta_1)+(\partial_0\theta_3)\sinh\theta_0+(\partial_3\theta_3)\cosh\theta_0\\
&\quad +2e^{\theta_0}\theta_A\theta_A(\partial_0\theta_3-\partial_3\theta_3)-2e^{\theta_0}\theta_A\partial_A\theta_3=f_6(\omega),\\
&7)\quad \Big(\partial_1\theta_0+2\theta_1(\partial_3\theta_0-\partial_0\theta_0)\Big)\cos\theta_3+\Big(\partial_2\theta_0+2\theta_2(\partial_3\theta_0\\
&\quad -\partial_0\theta_0)\Big)\sin\theta_3=f_7(\omega),\\
&8)\quad \Big(\partial_2\theta_0+2\theta_2(\partial_3\theta_0-\partial_0\theta_0)\Big)\cos\theta_3-\Big(\partial_1\theta_0+2\theta_1(\partial_3\theta_0\\
&\quad -\partial_0\theta_0)\Big)\sin\theta_3=f_8(\omega),\\
&9)\quad (\partial_0\omega)\cosh\theta_0+(\partial_3\omega)\sinh\theta_0-2e^{\theta_0}\theta_A\partial_A\omega+2e^{\theta_0}\\
&\quad \times\theta_A\theta_A(\partial_0\omega-\partial_3\omega)=f_9(\omega),\\
&10)\quad (\partial_3\omega)\cosh\theta_0+(\partial_0\omega)\sinh\theta_0-2e^{\theta_0}\omega_A\partial_A\omega+2e^{\theta_0}\\
&\quad \times\theta_A\theta_A(\partial_0\omega-\partial_3\omega)=f_{10}(\omega),\\
&11)\quad \Big(\partial_1\omega+2\theta_1(\partial_3\omega-\partial_0\omega)\Big)\cos\theta_3+\Big(\partial_2\omega+2\theta_2(\partial_3\omega\\
&\quad -\partial_0\omega)\Big)\sin\theta_3=f_{11}(\omega),\\
&12)\quad \Big(\partial_2\omega+2\theta_2(\partial_3\omega-\partial_0\omega)\Big)\cos\theta_3-\Big(\partial_1\omega+2\theta_1(\partial_3\omega\\
&\quad -\partial_0\omega)\Big)\sin\theta_3=f_{12}(\omega),
\end{aligned}
\tag{6.1.8}
$$

where $f_1(\omega),\dots,f_{12}(\omega)$ are arbitrary smooth real-valued functions.

Thus, the problem of construction of Ansätze (6.1.5) reducing the nonlinear Dirac equation (6.1.1) to systems of ODEs is equivalent to the one of integration of the over-determined system of PDEs (6.1.8). Let us emphasize that the above system is compatible because Poincaré-invariant Ansätze

obtained in Section 2.2 are contained in class (6.1.5).

Integration of the system of nonlinear PDEs (6.1.8) is substantially simplified if we utilize an equivalence relation which is introduced below.

First of all, we note that the class of Ansätze (6.1.5) is transformed into itself if we generate the spinor field (6.1.5) by the 8-parameter transformation group $G_8 \subset P(1,3)$ with the generators $P_\mu,\ J_{12},\ J_{03},\ J_{01}-J_{13},\ J_{02}-J_{23}$.

The above assertion is checked by a direct verification. Take, as an example, the one-parameter transformation group having the generator J_{03}. Applying formula (2.4.43) with $a=3$ to (6.1.5) we get

$$\begin{aligned}\psi(x) \quad = \quad & \exp\{\theta'_A(x')\gamma_A(\gamma_0+\gamma_3)\}\exp\{(1/2)\theta'_0(x')\gamma_0\gamma_3 \\ & +(1/2)\theta'_3(x')\gamma_1\gamma_2\}\varphi\Big(\omega'(x')\Big),\end{aligned}$$

where

$$\begin{aligned}& x'_0 = x_0\cosh\tau + x_3\sinh\tau, \quad x'_1 = x_1, \\ & x'_2 = x_2, \quad x'_3 = x_3\cosh\tau + x_0\sinh\tau, \\ & \theta'_0 = \theta_0+\tau, \quad \theta'_1 = \theta_1 e^{-\tau}, \quad \theta'_2 = \theta_2 e^{-\tau}, \\ & \theta'_3 = \theta_3, \quad \omega' = \omega.\end{aligned}$$

Consequently, the group G_8 induces in the space of variables $x,\ \theta_\mu(x)$, $\omega(x)$ some transformation group $\widetilde{G}_8$. It is not difficult to establish that $\widetilde{G}_8$ is the invariance group of system of PDEs (6.1.8).

Another transformation leaving the class of Ansätze (6.1.5) invariant is the following one:

$$\begin{aligned}& \theta_0 \to \theta_0 + g_0(\omega), \quad \theta_3 \to \theta_3 + g_3(\omega), \quad \omega \to g(\omega), \\ & \theta_1 \to \theta_1 + e^{-\theta_0}\Big(g_1(\omega)\cos\theta_3 - g_2(\omega)\sin\theta_3\Big), \\ & \theta_2 \to \theta_2 + e^{-\theta_0}\Big(g_2(\omega)\cos\theta_3 + g_1(\omega)\sin\theta_3\Big).\end{aligned} \tag{6.1.9}$$

That is why it is natural to introduce the following equivalence relation E. We say that solutions of system (6.1.5) $\theta_\mu(x),\ \omega(x)$ and $\theta'_\mu(x),\ \omega'(x)$ are equivalent if they can be transformed one into another by

1) a suitable transformation from the group $\widetilde{G}_8$, or

2) a suitable transformation of the form (6.1.9).

An easy check shows that E is indeed an equivalence relation. It divides the set of solutions of the system of PDEs under study into inequivalent classes which are described by the following assertion.

Theorem 6.1.1. *The general solution of system of PDEs (6.1.8) determined up to the equivalence relation E is given by one of the following formulae:*

1) $\theta_1 = \theta_2 = 0, \quad \theta_0 = \ln(x_0 + x_3), \quad \theta_3 = C\ln(x_0 + x_3), \quad \omega = x_0^2 - x_3^2;$

2) $\theta_A = -x_A\Big(2(x_0 + x_3)\Big)^{-1}, \quad \theta_0 = \ln(x_0 + x_3), \quad \theta_3 = C\ln(x_0 + x_3),$
$\omega = x_0^2 - x_1^2 - x_2^2 - x_3^2;$

3) $\theta_1 = 0, \quad \theta_2 = -x_2\Big(2(x_0 + x_3)\Big)^{-1}, \quad \theta_0 = \ln(x_0 + x_3),$
$\theta_3 = C\ln(x_0 + x_3), \quad \omega = x_0^2 - x_2^2 - x_3^2;$

4) $\theta_1 = \theta_2 = 0, \quad \theta_0 = 0, \quad \theta_3 = C_1(x_0 + x_3), \quad \omega = x_0 - x_3$
$+C_2(x_0 + x_3);$

5) $\theta_1 = \theta_2 = 0, \quad \theta_0 = Cx_1, \quad \theta_3 = 0, \quad \omega = Cx_1 + \ln(x_0 - x_3);$

6) $\theta_A = \partial_A W, \quad \theta_0 = 0, \quad \theta_3 = (C/2)(x_0 - x_3 + 4W), \quad \omega = x_0 + x_3,$
$W = \tau_1 z^2 + \tau_2 z + \tau_1^* z^{*2} + \tau_2^* z^* + \tau_3 z z^*,$
where $z = x_1 + ix_2$ *and the functions* $\tau_j(x)$ *are determined by one of the formulae* $a - c$ *given below*

a) $\tau_1 = C_2\Big(64C_2^2(x_0 + x_3)^2 - 1\Big)^{-1} e^{iC_1},$

$$\tau_2 = C_3 \exp\Bigg\{16C_2 \int^{(1/2)(x_0 + x_3)} (256C_2^2\xi^2 - 1)^{-1}\Big[-16C_2\xi + \cos\Big(2R_1(\xi) - C_1\Big)\Big]d\xi + iR_1\Big((1/2)(x_0 + x_3)\Big)\Bigg\},$$

$\tau_3 = 16C_2^2(x_0 + x_3)\Big(1 - 64C_2^2(x_0 + x_3)^2\Big)^{-1},$

$\dot{R}_1(\xi) = 16C_2(1 - 256C_2^2\xi^2)^{-1}\Big[16C_2\xi + \sin\Big(2R_1(\xi) - C_1\Big)\Big];$

b) $\tau_1 = \Big(16(x_0 + x_3)\Big)^{-1} e^{iC_1},$ (6.1.10)

$$\tau_2 = C_2 \exp\Bigg\{(1/2) \int^{(1/2)(x_0 + x_3)} \Big[\cos\Big(2R_2(\xi) - C_1\Big) - 1\Big]\xi^{-1}d\xi + iR_2\Big((1/2)(x_0 + x_3)\Big)\Bigg\}, \quad \tau_3 = -\Big(8(x_0 + x_3)\Big)^{-1},$$

$2\xi\dot{R}_2(\xi) + \sin\Big(2R_2(\xi) - C_1\Big) + 1 = 0;$

$$c)\quad \tau_1 = 0, \quad \tau_2 = (C_1 + iC_2)(x_0 + x_3)^{-1}, \quad \tau_3 = \Big(4(x_0 + x_3)\Big)^{-1};$$

7) $$\theta_A = (1/2)\dot{w}_A + C_2 \arctan(\tilde{x}_1/\tilde{x}_2)(\tilde{x}_1^2 + \tilde{x}_2^2)^{1/2} \times \exp\{-C_1 \arctan(\tilde{x}_1/\tilde{x}_2)\}\partial_A\Big(\arctan(\tilde{x}_1/\tilde{x}_2)\Big),$$
$$\theta_0 = C_1 \arctan(\tilde{x}_1/\tilde{x}_2), \quad \theta_3 = -\arctan(\tilde{x}_1/\tilde{x}_2), \quad \omega = \tilde{x}_1^2 + \tilde{x}_2^2;$$

8) $$\theta_A = (1/2)\dot{w}_A + (\tilde{x}_1^2 + \tilde{x}_2^2)^{1/2}\Big(C_1(x_0 + x_3)^{-1} \times \arctan(\tilde{x}_1/\tilde{x}_2) + w_3\Big)\partial_A\Big(\arctan(\tilde{x}_1/\tilde{x}_2)\Big),$$
$$\theta_0 = \ln(x_0 + x_3), \quad \theta_3 = -\arctan(\tilde{x}_1/\tilde{x}_2), \quad \omega = \tilde{x}_1^2 + \tilde{x}_2^2;$$

9) $$\theta_A = x_1 w_A + \partial_A\Big(U(z, x_0 + x_3) + U(z^*, x_0 + x_3)\Big), \quad z = x_1 + ix_2,$$
$$\theta_0 = \theta_3 = 0, \quad \omega = x_0 + x_3;$$

10) $$\theta_1 = (x_1 \sin w_2 - x_2 \cos w_2)\Big[\Big((1/2)\dot{w}_1 + Ce^{-w_1}\Big)\sin w_2 - (1/2)\dot{w}_2 \cos w_2\Big] + w_4 \sin w_2 + (1/2)\dot{w}_3 \cos w_2,$$
$$\theta_2 = (x_1 \sin w_2 - x_2 \cos w_2)\Big[-\Big((1/2)\dot{w}_1 + Ce^{-w_1}\Big)\cos w_2 - (1/2)\dot{w}_2 \sin w_2\Big] - w_4 \cos w_2 + (1/2)\dot{w}_3 \sin w_2,$$
$$\theta_0 = w_1, \quad \theta_3 = w_2, \quad \omega = x_1 \cos w_2 + x_2 \sin w_2 + w_3;$$

11) $$\theta_A = (1/2)\dot{w}_A, \quad \theta_0 = C(x_2 + w_2), \quad \theta_3 = 0, \quad \omega = x_1 + w_1.$$

In the above formulae $\tilde{x}_A = x_A + w_A$; $A = 1, 2$; w_1, w_2, w_3, w_4 *are arbitrary smooth real-valued functions of* $x_0 + x_3$; U *is an arbitrary analytic function of* z; C, C_1, C_2, C_3 *are arbitrary real constants.*

Proof. On introducing new independent variables $\xi = (1/2)(x_0 + x_3)$, $\eta = (1/2)(x_0 - x_3)$ we rewrite system (6.1.8) in the form

$$\begin{aligned}
&1)\quad \partial_\eta\theta_0 = f_1(\omega)e^{\theta_0},\\
&2)\quad \partial_\xi\theta_0 - 4\partial_A\theta_A - 4\theta_A\partial_A\theta_0 + 4f_1(\omega)e^{\theta_0}\theta_A\theta_A = f_2(\omega)e^{-\theta_0},\\
&3)\quad \partial_1\theta_3 = 2\theta_1 e^{\theta_0} f_1(\omega) + f_4(\omega)\cos\theta_3 - f_3(\omega)\sin\theta_3,\\
&4)\quad \partial_2\theta_3 = 2\theta_2 e^{\theta_0} f_1(\omega) + f_3(\omega)\cos\theta_3 + f_4(\omega)\sin\theta_3,\\
&5)\quad \partial_\eta\theta_3 = f_5(\omega)e^{\theta_0},\\
&6)\quad \partial_\xi\theta_3 + 4(\partial_1\theta_2 - \partial_2\theta_1) + 4f_5(\omega)e^{\theta_0}\theta_A\theta_A - 4\theta_A\partial_A\theta_3\\
&\qquad = f_6(\omega)e^{-\theta_0}, \qquad\qquad (6.1.11)\\
&7)\quad \partial_1\theta_0 = 2\theta_1 f_1(\omega)e^{\theta_0} + f_7(\omega)\cos\theta_3 - f_8(\omega)\sin\theta_3,\\
&8)\quad \partial_2\theta_0 = 2\theta_2 f_1(\omega)e^{\theta_0} + f_8(\omega)\cos\theta_3 + f_7(\omega)\sin\theta_3,
\end{aligned}$$

9) $\partial_\eta \omega = f_9(\omega) e^{\theta_0}$,

10) $\partial_\xi \omega - 4\theta_A \partial_A \omega + 4 f_9(\omega) e^{\theta_0} \theta_A \theta_A = f_{10}(\omega) e^{-\theta_0}$,

11) $\partial_1 \omega = 2\theta_1 f_9(\omega) e^{\theta_0} + f_{11}(\omega) \cos\theta_3 - f_{12}(\omega) \sin\theta_3$,

12) $\partial_2 \omega = 2\theta_2 f_9(\omega) e^{\theta_0} + f_{12}(\omega) \cos\theta_3 + f_{11}(\omega) \sin\theta_3$.

Now we see that the above system contains a subsystem of PDEs 1, 5, 9

$$\partial_\eta \theta_0 = f_1(\omega) e^{\theta_0}, \quad \partial_\eta \theta_3 = f_5(\omega) e^{\theta_0}, \quad \partial_\eta \omega = f_9(\omega) e^{\theta_0},$$

which can be considered as a system of ODEs with respect to the variable η. Transforming $\theta_0,\ \theta_3,\ \omega$ according to (6.1.9) we can put $f_1 f_9 = 0$. With this remark the above system is easily integrated. Its general solution determined up to the equivalence relation E is given by one of the following formulae:

I. under $f_1 = f_5 = f_9 = 0$,
$\theta_0 = F_1, \quad \theta_3 = F_2, \quad \omega = F_3$;

II. under $f_1 = f_5 = 0,\ f_9 \neq 0$,
$\theta_0 = \ln F_1,\ \omega = \eta F_1 + F_2,\ \theta_3 = F_3$;

III. under $f_9 = 0,\ f_1 \neq 0$,
$\theta_0 = -\ln(\eta + F_2), \quad \omega = F_1, \quad \theta_3 = f_5(F_1)\ln(\eta + F_2) + F_3$;

IV. under $f_1 = f_9 = 0,\ f_5 \neq 0$,
$\theta_0 = -\ln F_2, \quad \theta_3 = F_2^{-1} f_5(F_1)\eta + F_3, \quad \omega = F_1$,

where $F_1,\ F_2,\ F_3$ are arbitrary smooth real-valued functions of $\xi,\ x_1,\ x_2$.

Thus, to prove the theorem we have to consider four inequivalent cases I–IV. We will integrate system of PDEs (6.1.11) in the case $f_1 = f_5 = f_9 = 0$, the remaining cases are handled in an analogous way.

When proving the theorem, we will use essentially the following assertion.

Lemma 6.1.1. *General solution of system of PDEs*

$$\begin{aligned}
\partial_1 u &= A_1(u)\cos v - A_2(u)\sin v,\\
\partial_2 u &= A_2(u)\cos v + A_1(u)\sin v,\\
\partial_1 v &= B_1(u)\cos v - B_2(u)\sin v,\\
\partial_2 v &= B_2(u)\cos v + B_1(u)\sin v,
\end{aligned}$$

determined up to the equivalence relation

$$u \to h_1(u), \quad v \to v + h_2(u), \quad h_i \in C^1(\mathbb{R}^1, \mathbb{R}^1)$$

is given by one of the formulae

$$
\begin{aligned}
&1)\quad u = (x_1 + w_1)^2 + (x_2 + w_2)^2,\\
&\quad v = \arctan\Big((x_1 + w_1)(x_2 + w_2)^{-1}\Big);\\
&2)\quad u = x_1 \cos w_2 + x_2 \sin w_2 + w_1,\ \ v = w_2;\\
&3)\quad u = w_1,\ \ v = w_2.
\end{aligned}
\tag{6.1.12}
$$

Here $w_1,\ w_2$ are arbitrary smooth real-valued functions of ξ.

Proof of the above assertion is carried out with the help of rather simple but very cumbersome computations, therefore it is omitted.

Substituting $\theta_0 = F_1(\xi, x_1, x_2),\ \theta_3 = F_2(\xi, x_1, x_2),\ \omega = F_3(\xi, x_1, x_2)$ into system (6.1.11) we have

$$
\begin{aligned}
&1)\quad \partial_\xi F_1 - 4\partial_A\theta_A - 4\theta_A\partial_A F_1 = f_2 e^{-F_1},\\
&2)\quad \partial_1 F_2 = f_4 \cos F_2 - f_3 \sin F_2,\\
&3)\quad \partial_2 F_2 = f_3 \cos F_2 + f_4 \sin F_2,\\
&4)\quad \partial_\xi F_2 + 4(\partial_1\theta_2 - \partial_2\theta_1) - 4\theta_A\partial_A F_2 = f_6 e^{-F_1},\\
&5)\quad \partial_1 F_1 = f_7 \cos F_2 - f_8 \sin F_2,\\
&6)\quad \partial_2 F_1 = f_8 \cos F_2 + f_7 \sin F_2,\\
&7)\quad \partial_\xi F_3 - 4\theta_A\partial_A F_3 = f_{10} e^{-F_1},\\
&8)\quad \partial_1 F_3 = f_{11} \cos F_2 - f_{12} \sin F_2,\\
&9)\quad \partial_2 F_3 = f_{12} \cos F_2 + f_{11} \sin F_2,
\end{aligned}
\tag{6.1.13}
$$

where $f_2, \ldots, f_{12}$ are arbitrary smooth real-valued functions of F_3.

According to Lemma 6.1.1 a subsystem of equations 2, 3, 8, 9 has three inequivalent classes of solutions given by formulae (6.1.12).

Case 1. $F_2 = -\arctan\Big((x_1 + w_1)(x_2 + w_2)^{-1}\Big), \quad F_3 = (x_1 + w_1)^2 + (x_2 + w_2)^2.$

Substitution of the above expressions into the fifth and sixth equations of system (6.1.13) yields the following system of PDEs for the function F_1:

$$
\begin{aligned}
\partial_1 F_1 &= \tilde{x}_2 \tilde{f}_7(\tilde{x}_1^2 + \tilde{x}_2^2) + \tilde{x}_1 \tilde{f}_8(\tilde{x}_1^2 + \tilde{x}_2^2),\\
\partial_2 F_1 &= \tilde{x}_2 \tilde{f}_8(\tilde{x}_1^2 + \tilde{x}_2^2) - \tilde{x}_1 \tilde{f}_8(\tilde{x}_1^2 + \tilde{x}_2^2),
\end{aligned}
\tag{6.1.14}
$$

where $\tilde{x}_A = x_A + w_A,\ A = 1, 2.$

Taking into account the compatibility condition $\partial_1(\partial_2 F_1) = \partial_2(\partial_1 F_1)$ we have $\dot{\tilde{f}}_7 = 0$ or $\tilde{f}_7 = C_1 = \text{const}$. Hence it follows that up to the equivalence relation E the general solution of system (6.1.14) can be represented in the form

$$F_1 = C_1 \arctan(\tilde{x}_1/\tilde{x}_2) + w_3(\xi), \quad w_3 \in C^1(\mathbb{R}^1, \mathbb{R}^1).$$

From the seventh equation of system (6.1.13) it follows that functions $\theta_1,\ \theta_2$ satisfy the equality

$$\tilde{x}_A(\dot{w}_A - 4\theta_A) = f_{10}(\tilde{x}_1^2 + \tilde{x}_2^2) \exp\{-C_1 \arctan(\tilde{x}_1/\tilde{x}_2) - w_3\},$$

whose general solution reads

$$\theta_A = (1/4)\dot{w}_A + W(\xi, x_1, x_2)\partial_A\Big(\arctan(\tilde{x}_1/\tilde{x}_2)\Big), \quad A = 1, 2.$$

Here W is an arbitrary smooth real-valued function.

Substituting the above results into the first and fourth equations of system (6.1.13) we arrive at the following system of two PDEs for W:

$$\begin{aligned} &\tilde{x}_A\partial_A W = W + \alpha_1 \exp\{-C_1 \arctan(\tilde{x}_1/\tilde{x}_2) - w_3\}, \\ &(\tilde{x}_2\partial_1 - \tilde{x}_1\partial_2)W = -C_1 W + (1/4)\dot{w}_3(\tilde{x}_1^2 + \tilde{x}_2^2) \\ &\quad +\alpha_2 \exp\{-C_1 \arctan(\tilde{x}_1/\tilde{x}_2) - w_3\}, \end{aligned} \tag{6.1.15}$$

where $\alpha_A = \alpha_A(\tilde{x}_1^2 + \tilde{x}_2^2)$. Integration of system of linear PDEs (6.1.15) yields two inequivalent classes of solutions

<u>under $C_1 \neq 0$</u>

$$\begin{aligned} W \ &= \ \Big(C_3 + C_2 \arctan(\tilde{x}_1/\tilde{x}_2)\Big)(\tilde{x}_1^2 + \tilde{x}_2^2)^{1/2} \\ &\quad \times \exp\{-C_1 \arctan(\tilde{x}_1/\tilde{x}_2)\}, \quad w_3 = 0; \end{aligned}$$

<u>under $C_1 = 0$</u>

$$W \ = \ \Big(w_0(\xi) + C\xi^{-1} \arctan(\tilde{x}_1/\tilde{x}_2)\Big)(\tilde{x}_1^2 + \tilde{x}_2^2)^{1/2}, \quad w_3 = \xi.$$

Here $C,\ C_1,\ C_2,\ C_3$ are real constants, $w_0 \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function.

Substituting the results obtained into the corresponding expressions for $\theta_\mu,\ \omega$ and returning to the initial independent variables x_μ we get up to the equivalence relation E the formulae 8, 9 from (6.1.10).

Case 2. $F_2 = w_2(\xi), \quad F_3 = w_3(\xi)$.

Up to the equivalence relation E we can choose $F_3 = \xi,\ F_2 = 0$. Substitution of these expressions into (6.1.13) gives rise to the following system of PDEs for $F_1,\ \theta_1,\ \theta_2$:

$$\begin{aligned} &1)\quad \partial_\xi F_1 - 4\theta_A \partial_A F_1 - 4\partial_A \theta_A = f_2(\xi) e^{-F_1},\\ &2)\quad \partial_2\theta_1 - \partial_1\theta_2 = f_6(\xi) e^{-F_1},\\ &3)\quad \partial_1 F_1 = f_7(\xi),\\ &4)\quad \partial_2 F_1 = f_8(\xi),\\ &5)\quad 1 = f_{10}(\xi) e^{-F_1}. \end{aligned} \tag{6.1.16}$$

From the last three equations we conclude that within the equivalence relation $F_1 = 0$. Integrating the remaining equations and returning to the initial independent variables we obtain within the equivalence relation E formulae 9 from (6.1.10).

Case 3. $F_2 = w_2(\xi), \quad F_3 = x_1 \cos w_2(\xi) + x_2 \sin w_2(\xi) + w_3(\xi)$.

Substitution of the above expressions into equations 1, 4–7 from (6.1.13) gives rise to the over-determined system of PDEs for functions $F_1,\ \theta_1,\ \theta_2$

$$\begin{aligned} &1)\quad 4\partial_A\theta_A = \partial_\xi F_1 - 4\theta_A\partial_A F_1 + f_2 e^{-F_1},\\ &2)\quad 4(\partial_2\theta_1 - \partial_1\theta_2) = \dot{w}_2 + f_6 e^{-F_1},\\ &3)\quad \partial_1 F_1 = f_7 \cos w_2 - f_8 \sin w_2,\\ &4)\quad \partial_2 F_1 = f_8 \cos w_2 + f_7 \sin w_2,\\ &5)\quad \dot{w}_2(x_2\cos w_2 - x_1 \sin w_2) + \dot{w}_3\\ &\qquad -4(\theta_1 \cos w_2 + \theta_2 \sin w_2) = f_{10} e^{-F_1}, \end{aligned} \tag{6.1.17}$$

where $f_2,\ f_6,\ f_7,\ f_8,\ f_{10}$ are arbitrary smooth functions of $x_1 \cos w_2 + x_2 \sin w_2 + w_3$.

The necessary and sufficient compatibility condition of a subsystem of equations 3, 4 reads $\partial_1(\partial_2 F_1) = \partial_2(\partial_1 F_1)$, whence it follows that $f_8 = C_1 =$ const. Substituting $f_8 = C_1$ into equations 3, 4 from (6.1.17) and integrating the equations obtained we have

$$F_1 = C_1(x_2 \cos w_2 - x_1 \sin w_2) + w_1(\xi), \quad w_1 \in C^1(\mathbb{R}^1, \mathbb{R}^1).$$

With account of the above formula system (6.1.17) is rewritten in the following way:

$$1)\quad \partial_A\theta_A = -(1/4)C_1\dot{w}_2(x_1\cos w_2 + x_2 \sin w_2) + (1/4)\dot{w}_1$$

$$
\begin{aligned}
&\quad +C_1(\theta_1 \sin w_2 - \theta_2 \cos w_2) + f_2 \exp\{-C_1(x_2 \cos w_2 \\
&\quad -x_1 \sin w_2) - w_1\}, \\
&2)\quad \partial_2\theta_1 - \partial_1\theta_2 = (1/4)\dot{w}_2 + f_6 \exp\{-C_1(x_2 \cos w_2 \\
&\quad -x_1 \sin w_2) - w_1\}, \\
&3)\quad \theta_1 \cos w_2 + \theta_2 \sin w_2 = (1/4)\dot{w}_2(x_2 \cos w_2 - x_1 \sin w_2) \\
&\quad +(1/4)\dot{w}_3 + f_{10} \exp\{-C_1(x_2 \cos w_2 - x_1 \sin w_2) - w_1\}.
\end{aligned}
\tag{6.1.18}
$$

Integrating equations (6.1.18) we get up to the equivalence relation E the formulae 10, 11 from (6.1.10) under $C_1 = 0$ and $C_1 \neq 0$, respectively. The theorem is proved. $\triangleright$

Choosing in an appropriate way parameters and arbitrary functions we can obtain from (6.1.5) and (6.1.10) Ansätze invariant under the $P(1,3)$ non-conjugate three-dimensional subalgebras of the algebra $AP(1,3)$ constructed in Section 2.2. Hence it follows, in particular, that the classical Lie approach gives no complete description of Ansätze reducing nonlinear PDE (6.1.1) to ODEs. Additional possibilities of reduction of equation (6.1.1) are the consequence of its conditional symmetry. To become convinced of this fact we will construct involutive sets of the first-order differential operators

$$Q_a = \xi_{a\mu}(x)\partial_\mu + \eta_a(x), \quad a = 1, 2, 3,$$

where $\xi_{a\mu}(x)$ are real-valued scalar functions, $\eta_a(x)$ are (4×4)-matrices, such that Ansätze (6.1.5), (6.1.10) are invariant with respect to these operators. Then, we will show that the nonlinear Dirac equation (6.1.1) is conditionally-invariant with respect to so obtained involutive sets of differential operators.

According to Definition 1.5.2 Ansatz (6.1.5) is invariant with respect to the involutive set of operators $Q_1,\ Q_2,\ Q_3$ if the conditions

$$Q_a\psi(x) \equiv Q_a\Big(A(x)\varphi(\omega)\Big) = 0,\ a = 1, 2, 3, \tag{6.1.19}$$

where $A(x) = \exp\{\gamma_A\theta_A(\gamma_0 + \gamma_3)\}\exp\{(1/2)\theta_0\gamma_0\gamma_3 + (1/2)\theta_3\gamma_1\gamma_2\}$, hold with an arbitrary four-component function $\varphi(\omega)$.

Equating coefficients of $\varphi(\omega)$ and $\dot{\varphi}(\omega)$ in the left-hand side of (6.1.19) to zero we get

$$\xi_{a\mu}(x)\partial_\mu\omega(x) = 0,\ a = 1, 2, 3, \tag{6.1.20}$$

$$\eta_a(x) = -\Big(\xi_{a\mu}(x)\partial_\mu A\Big)A^{-1},\ a = 1, 2, 3. \tag{6.1.21}$$

Thus, to obtain the involutive set of operators O_a such that the Ansatz (6.1.5) is invariant with respect to it we have

- to solve equations (6.1.20) which should be considered as a system of linear algebraic equations with respect to $\xi_{a\mu}(x),\ a = 1, 2, 3,\ \mu = 0, \ldots, 3$;
- to get explicit expressions for $\eta_a,\ a = 1, 2, 3$ from (6.1.21).

On solving equations (6.1.20), (6.1.21) for each class of functions $\theta_\mu(x)$, $\omega(x)$ from (6.1.10) we obtain the following sets of operators Q_a:

$$
\begin{aligned}
1)\quad & Q_1 = \partial_1,\ Q_2 = \partial_2,\ Q_3 = x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3 - C\Big(x_2\partial_1\\
&\quad -x_1\partial_2 + (1/2)\gamma_1\gamma_2\Big);\\
2)\quad & Q_1 = (x_0 + x_3)\partial_1 + x_1(\partial_0 - \partial_3) + (1/2)\gamma_1(\gamma_0 + \gamma_3),\\
& Q_2 = (x_0 + x_3)\partial_2 + x_2(\partial_0 - \partial_3) + (1/2)\gamma_2(\gamma_0 + \gamma_3),\\
& Q_3 = x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3 - C\Big(x_2\partial_1 - x_1\partial_2 + (1/2)\gamma_1\gamma_2\Big);\\
3)\quad & Q_1 = \partial_1,\ Q_2 = (x_0 + x_3)\partial_2 + x_2(\partial_0 - \partial_3) + (1/2)\gamma_2(\gamma_0 + \gamma_3),\\
& Q_3 = x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3 - C\Big(x_2\partial_1 - x_1\partial_2 + (1/2)\gamma_1\gamma_2\Big);\\
4)\quad & Q_1 = \partial_1,\ Q_2 = \partial_2,\ Q_3 = (1 - C_2)\partial_0 + (1 + C_2)\partial_3\\
&\quad -2C_1\Big(x_2\partial_1 - x_1\partial_2 + (1/2)\gamma_1\gamma_2\Big);\\
5)\quad & Q_1 = \partial_0 + \partial_3,\ Q_2 = \partial_2,\\
& Q_3 = \partial_1 + C\Big(x_0\partial_3 + x_3\partial_0 - (1/2)\gamma_0\gamma_3\Big);\\
6)\quad & Q_A = \partial_A - \gamma_B(\partial_B\partial_A W)(\gamma_0 + \gamma_3) - 2C(\partial_A W)\Big(\gamma_1\gamma_2\\
&\quad +2(\gamma_1\partial_2 W - \gamma_2\partial_1 W)(\gamma_0 + \gamma_3)\Big),\ A = 1, 2,\\
& Q_3 = \partial_0 - \partial_3 - C\gamma_1\gamma_2 - 2C(\gamma_1\partial_2 W - \gamma_2\partial_1 W)(\gamma_0 + \gamma_3);\\
7)\quad & Q_1 = \partial_0 - \partial_3,\ Q_2 = \tilde{x}_1\partial_2 - \tilde{x}_2\partial_1 - (1/2)(\gamma_1\gamma_2 - C_1\gamma_0\gamma_3)\\
&\quad +C_1(\tilde{x}_1^2 + \tilde{x}_2^2)^{-1/2}\exp\{-C_1\arctan(\tilde{x}_1/\tilde{x}_2)\}\\
&\quad \times(\gamma_2\tilde{x}_1 - \gamma_1\tilde{x}_2)(\gamma_0 + \gamma_3) + (1/2)(\gamma_1\dot{w}_2 - \gamma_2\dot{w}_1)\\
&\quad \times(\gamma_0 + \gamma_3) - (C_1/2)\gamma_A\dot{w}_A(\gamma_0 + \gamma_3),\\
& Q_3 = \partial_0 + \partial_3 - 2\dot{w}_A\partial_A - \gamma_A\ddot{w}_A(\gamma_0 + \gamma_3);\\
8)\quad & Q_1 = \partial_0 - \partial_3,\ Q_2 = \tilde{x}_1\partial_2 - \tilde{x}_2\partial_1 - (1/2)\gamma_1\gamma_2 - C_1(\tilde{x}_1^2\\
&\quad +\tilde{x}_2^2)^{-1/2}(x_0 + x_1)^{-1}(\tilde{x}_1\gamma_2 - \tilde{x}_2\gamma_1)(\gamma_0 + \gamma_3)\\
&\quad -(1/2)(\gamma_1\dot{w}_2 - \gamma_2\dot{w}_1)(\gamma_0 + \gamma_3),\\
& Q_3 = x_0\partial_3 + x_3\partial_0 - (x_0 + x_3)\dot{w}_A\partial_A - (1/2)\gamma_0\gamma_3\\
&\quad -(1/2)\gamma_A\Big(\dot{w}_A + (x_0 + x_3)\ddot{w}_A\Big)(\gamma_0 + \gamma_3) - (\tilde{x}_1^2 + \tilde{x}_2^2)^{-1/2}
\end{aligned}
\tag{6.1.22}
$$

$$\times\Big(w_3 + (x_0 + x_3)\dot{w}_3\Big)(\gamma_1\tilde{x}_2 - \gamma_2\tilde{x}_1)(\gamma_0 + \gamma_3);$$

9) $$Q_A = \partial_A - \gamma_B\Big[\partial_B\partial_A\Big(U(x_1 + ix_2,\, x_0 + x_3) + U(x_1 - ix_2,\, x_0 + x_3)\Big)\Big](\gamma_0 + \gamma_3) + \delta_{A1}\gamma_B W_B \times(\gamma_0 + \gamma_3),\ A = 1, 2,\ Q_3 = \partial_0 - \partial_3;$$

10) $$Q_1 = \partial_0 - \partial_3,\ Q_2 = (\sin w_2)\partial_1 - (\cos w_2)\partial_2 - \Big[\gamma_1\Big[\Big((1/2)\dot{w}_1 + Ce^{-w_1}\Big)\sin w_2 - (1/2)\dot{w}_2 \times\cos w_2\Big] - \gamma_2\Big[\Big((1/2)\dot{w}_1 + Ce^{-w_1}\Big)\cos w_2 + (1/2)\dot{w}_2 \sin w_2\Big]\Big](\gamma_0 + \gamma_3),\ Q_3 = \dot{w}_2(x_1\partial_2 - x_2\partial_1) + \dot{w}_3\Big((\cos w_2)\partial_1 + (\sin w_2)\partial_2\Big) + (1/2)(\partial_0 + \partial_3) - (x_1 \sin w_2 - x_2 \cos w_2)\Big((1/2)(\ddot{w}_1 + \dot{w}_1^2)(\gamma_1 \sin w_2 - \gamma_2 \cos w_2) - (1/2)(\ddot{w}_2 + \dot{w}_1\dot{w}_2)(\gamma_1 \cos w_2 + \gamma_2 \sin w_2)\Big) \times(\gamma_0 + \gamma_3) - (\dot{w}_4 + \dot{w}_1 w_4)(\gamma_1 \sin w_2 - \gamma_2 \cos w_2)(\gamma_0 + \gamma_3) - (1/2)(\ddot{w}_3 + \dot{w}_1\dot{w}_3)(\gamma_1 \cos w_2 + \gamma_2 \sin w_2)(\gamma_0 + \gamma_3) - (1/2)\dot{w}_2\gamma_1\gamma_2 - (1/2)\dot{w}_1\gamma_0\gamma_3,$$

11) $$Q_1 = \partial_0 - \partial_3,\ Q_2 = \partial_2 - (C/2)\gamma_0\gamma_3 - (C/2)\gamma_A\dot{w}_A(\gamma_0 + \gamma_3), \quad Q_3 = -\dot{w}_1\partial_1 + (1/2)(\partial_0 + \partial_3) - (1/2)\gamma_A\ddot{w}_A(\gamma_0 + \gamma_3) - (C/2)\dot{w}_2\gamma_0\gamma_3 - (C/2)\dot{w}_2\gamma_A\dot{w}_A(\gamma_0 + \gamma_3).$$

Analyzing the above formulae we come to a conclusion that only the operators 1–5 from (6.1.22) are linear combinations of the generators of the Poincaré group $P(1, 3)$ (1.1.20). The remaining triplets of operators cannot be represented as linear combinations of operators (1.1.20). Consequently, Ansätze 6–11 from (6.1.10) are not invariant with respect to three-parameter subgroups of the group $P(1, 3)$ and cannot, in principle, be constructed within the framework of the Lie approach. They correspond to conditional symmetry of the nonlinear Dirac equation (6.1.1).

Let us consider as an example the eighth triplet of operators $Q_1,\ Q_2,\ Q_3$. Rather tiresome computations yield the following relations:

$$\tilde{Q}_1 L = 0,$$
$$\tilde{Q}_2 L = -2iC_1(\tilde{x}_1^2 + \tilde{x}_2^2)^{-1/2}(x_0 + x_3)\Big((\gamma_0 + \gamma_3)Q_2\psi$$

$$
\begin{aligned}
&+(\gamma_1\tilde{x}_2-\gamma_2\tilde{x}_1)Q_1\psi\Big)+i(\gamma_1\dot{w}_2-\gamma_2\dot{w}_1)Q_1\psi+\Big(C_1(\tilde{x}_1^2\\
&+\tilde{x}_2^2)^{-1/2}(x_0+x_3)^{-1}(\tilde{x}_1\gamma_2-\tilde{x}_2\gamma_1)(\gamma_0+\gamma_3)\\
&+(1/2)\gamma_1\gamma_2+(1/2)(\gamma_1\dot{w}_2-\gamma_2\dot{w}_1)(\gamma_0+\gamma_3)\Big)L, \qquad (6.1.23)\\
\tilde{Q}_3L&=2i\Big(w_3+(x_0+x_3)\dot{w}_3\Big)(\tilde{x}_1^2+\tilde{x}_2^2)^{-1/2}\Big((\gamma_0+\gamma_3)Q_2\psi\\
&+(\gamma_1\tilde{x}_2-\gamma_2\tilde{x}_1)Q_1\psi\Big)+\Big[(1/2)\gamma_0\gamma_3+(1/2)\gamma_A\Big(\dot{w}_A\\
&+(x_0+x_3)\ddot{w}_A\Big)(\gamma_0+\gamma_3)+(\tilde{x}_1^2+\tilde{x}_2^2)^{-1/2}\Big(w_3+(x_0+x_3)\\
&\times\dot{w}_3\Big)(\gamma_1\tilde{x}_2-\gamma_2\tilde{x}_1)(\gamma_0+\gamma_3)\Big]L.\\
[Q_1,\,Q_2]&=[Q_2,\,Q_3]=0,\quad [Q_3,\,Q_1]=Q_1,
\end{aligned}
$$

In (6.1.23) we designate by the symbol $\tilde{Q}_a$ the first prolongation of operator Q_a, $L=i\gamma_\mu\psi_{x_\mu}-\tilde{f}_1\psi-\tilde{f}_2\gamma_4\psi$.

Thus, the nonlinear Dirac equation (6.1.1) is conditionally-invariant with respect to the involutive set of operators $Q_1,\ Q_2,\ Q_3$.

Substitution of the Ansätze (6.1.5), (6.1.10) into (6.1.1) gives rise to the following systems of ODEs for the four-component function $\varphi=\varphi(\omega)$:

1) $(1/2)(\gamma_0+\gamma_3)(1+C\gamma_1\gamma_2)\varphi+\Big(\gamma_0-\gamma_3+\omega(\gamma_0+\gamma_3)\Big)\dot{\varphi}=R,$

2) $(1/2)(\gamma_0+\gamma_3)(3+C\gamma_1\gamma_2)\varphi+\Big(\gamma_0-\gamma_3+\omega(\gamma_0+\gamma_3)\Big)\dot{\varphi}=R,$

3) $(1/2)(\gamma_0+\gamma_3)(2+C\gamma_1\gamma_2)\varphi+\Big(\gamma_0-\gamma_3+\omega(\gamma_0+\gamma_3)\Big)\dot{\varphi}=R,$

4) $(C_1/2)(\gamma_0+\gamma_3)\gamma_4\varphi+\Big(C_2(\gamma_0+\gamma_3)+\gamma_0-\gamma_3\Big)\dot{\varphi}=R,$

5) $(1/2)\gamma_2\gamma_4\varphi+\Big(C\gamma_1+e^{-\omega}(\gamma_0+\gamma_3)\Big)\dot{\varphi}=R,$

6) $-\Big(C(\gamma_0-\gamma_3)\gamma_4+4\tau_3(\omega)(\gamma_0+\gamma_3)+8C|\tau_2(\omega)|^2$
$\times(\gamma_0+\gamma_3)\gamma_4\Big)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,$

7) $\omega^{-1/2}\Big((1/2)\gamma_2-C_2(\gamma_0+\gamma_3)+(C_1/2)\gamma_2\gamma_4\Big)\varphi+2\omega^{1/2}\gamma_2\dot{\varphi}=R,$

8) $(1/2)\Big((1-2C_1\omega^{-1/2})(\gamma_0+\gamma_3)+\omega^{-1/2}\gamma_2\Big)\varphi+2\omega^{1/2}\gamma_2\dot{\varphi}=R,$

9) $(\gamma_0+\gamma_3)\Big(w_2(\omega)\gamma_4-w_1(\omega)\Big)\varphi+(\gamma_0+\gamma_3)\dot{\varphi}=R,$

10) $-C(\gamma_0+\gamma_3)\varphi+\gamma_1\dot{\varphi}=R,$

11) $-C\gamma_1\gamma_4\varphi+\gamma_1\dot{\varphi}=R,$

where $R = -i\Big(\tilde{f}_1(\bar{\varphi}\varphi, \bar{\varphi}\gamma_4\varphi) + \tilde{f}_2(\bar{\varphi}\varphi, \bar{\varphi}\gamma_4\varphi)\gamma_4\Big)\varphi$.

It is important to emphasize a very important difference between Poincaré-invariant Ansätze for the spinor field and conditionally-invariant Ansätze given in (6.1.10). As it was said above, $P(1,3)$-invariant Ansätze for the spinor field reduce any Poincaré-invariant spinor equation to systems of ODEs, provided the generators of the Poincaré group have the form (1.1.20). But for Ansätze (6.1.10) it is not the case. Each specific equation gives rise to a specific system of PDEs for functions $\theta_\mu,\ \omega$. This means that the approach suggested makes it possible to take into account a structure of solutions of the equation under study more precisely than the Lie approach does.

It is worth noting that the formula (6.1.5) can be easily adapted to the case of a field with an arbitrary spin s. Let us rewrite it in the following way:

$$\psi(x) = \exp\{2\theta_A(S_{0A} - S_{A3})\}\exp\{\theta_0 S_{03} + \theta_3 S_{12}\}\varphi(\omega), \qquad (6.1.24)$$

where $S_{\mu\nu} = (1/4)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. Ansatz (6.1.24) can be applied to reduce any Poincaré-invariant equation (by means of the method described above) provided it admits the group $P(1,3)$ with the following generators:

$$P_\mu = \partial^\mu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}.$$

Here $S_{\mu\nu}$ are constant matrices of the corresponding dimension satisfying the commutation relations of the Lie algebra $AO(1,3)$.

2. Non-Lie reduction of spinor equations invariant under the extended Poincaré group. We look for solutions of the nonlinear $\tilde{P}(1,3)$-invariant equations

$$\Big\{i\gamma_\mu\partial_\mu - (\bar{\psi}\psi)^{1/2k}\Big[g_1\Big(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}\Big) + g_2\Big(\bar{\psi}\psi(\bar{\psi}\gamma_4\psi)^{-1}\Big)\gamma_4\Big]\Big\}\psi = 0 \quad (6.1.25)$$

in the form

$$\psi(x) = \exp\{\theta_0 + \gamma_A\theta_A(\gamma_0 + \gamma_3)\}\varphi(\omega), \qquad (6.1.26)$$

where $\theta_0,\ \theta_1,\ \theta_2,\ \omega$ are arbitrary smooth real-valued functions of $x_0 + x_3,\ x_1,\ x_2;\ \varphi$ is an arbitrary complex-valued four-component function.

Substituting the Ansatz (6.1.26) into (6.1.25) and multiplying the expression obtained by the matrix $\exp\{-\theta_0 - \theta_A\gamma_A(\gamma_0 + \gamma_3)\}$ yield

$$iR_1(x)\varphi + iR_2(x)\dot{\varphi} = (\bar{\varphi}\varphi)^{1/2k}(g_1 + g_2\gamma_4)\varphi,$$

where

$$\begin{aligned}
g_A &= g_A\Big(\bar\varphi\varphi(\bar\varphi\gamma_4\varphi)^{-1}\Big), \quad A = 1,2,\\
R_1 &= (\gamma_0+\gamma_3)\partial_\xi\theta_0 + \gamma_A\partial_A\theta_0 + \gamma_A\gamma_B\partial_A\theta_B(\gamma_0+\gamma_3) - 2\theta_A\partial_A\theta_0(\gamma_0+\gamma_3),\\
R_2 &= (\gamma_0+\gamma_3)(\partial_\xi\omega - 2\theta_A\partial_A\omega) + \gamma_A\partial_A\omega
\end{aligned}$$

(as earlier, the notation $\xi = x_0 + x_3$ is used).

Consequently, Ansatz (6.1.26) reduces the initial equation (6.1.25) to a system of ODEs if there exist such (4×4)-matrices $G_1(\omega),\ G_2(\omega)$ that $R_A(x) = G_A(\omega),\ A = 1,2$. Hence we get the system of nonlinear PDEs for unknown functions $\theta_0,\ \theta_1,\ \theta_2,\ \omega$

$$\begin{aligned}
&1)\quad (\partial_\xi - 2\theta_A\partial_A)\theta_0 - \partial_A\theta_A = f_1(\omega)\exp\{\theta_0 k^{-1}\},\\
&2)\quad \partial_1\theta_0 = f_2(\omega)\exp\{\theta_0 k^{-1}\},\\
&3)\quad \partial_2\theta_0 = f_3(\omega)\exp\{\theta_0 k^{-1}\},\\
&4)\quad \partial_2\theta_1 - \partial_1\theta_2 = f_4(\omega)\exp\{\theta_0 k^{-1}\},\\
&5)\quad (\partial_\xi - 2\theta_A\partial_A)\omega = f_5(\omega)\exp\{\theta_0 k^{-1}\},\\
&6)\quad \partial_1\omega = f_6(\omega)\exp\{\theta_0 k^{-1}\},\\
&7)\quad \partial_2\omega = f_7(\omega)\exp\{\theta_0 k^{-1}\}.
\end{aligned} \tag{6.1.27}$$

In (6.1.27) $f_1,\dots,f_7$ are arbitrary smooth real-valued functions.

Solutions of the above system of nonlinear PDEs are looked for up to the equivalence relation E which is introduced in the following way. We say that the solutions of equations (6.1.27) $\theta_0(x),\ \theta_A(x),\ \omega(x)$ and $\theta_0'(x),\ \theta_A'(x),\ \omega'(x)$ are equivalent if they are transformed one into another by

1) a suitable transformation from the group $\widetilde G_8$, which is induced in the space of variables $x,\ \theta_0(x),\ \theta_A(x),\ \omega(x)$ by the action of the transformation group $G_8 \subset \widetilde P(1,3)$ with generators P_μ, $J_{01}-J_{13}$, $J_{02}-J_{23}$, J_{12}, D on Ansatz (6.1.26), or

2) a suitable transformation of the form

$$\omega \to h(\omega), \quad \theta_0 \to \theta_0 + h_0(\omega), \quad \theta_A \to \theta_A + h_A(\omega), \tag{6.1.28}$$

where $\{h, h_0, h_1, h_2\} \subset C^1(\mathbb{R}^1,\mathbb{R}^1)$.

Due to the fact that system (6.1.27) is over-determined we have succeeded in constructing its general solution. Up to the equivalence relation E it is given by one of the formulae

$$1)\quad \theta_0 = k\ln w_1, \quad \theta_1 = (2w_1)^{-1}(\dot w_1 x_1 + \dot w_2),$$

$$\begin{aligned}
&\theta_2 = (2w_1)^{-1}\Big((2k-1)\dot{w}_1 x_2 + w_3\Big), \quad \omega = w_1 x_1 + w_2;\\
2)\quad &\theta_0 = -k\ln(x_0 + w_1), \quad \theta_A = w_3\Big((x_1+w_1)^2\\
&\quad +(x_2+w_2)^2\Big)^{k-1}(x_A + w_A) + (1/2)\dot{w}_A, \quad A = 1,2, \qquad (6.1.29)\\
&\omega = (x_1+w_1)(x_2+w_2)^{-1};\\
3)\quad &\theta_0 = 0, \quad \omega = x_0 + x_3, \quad \theta_A = \partial_A\Big(U(x_1 + ix_2,\, x_0+x_3)\\
&\quad +U(x_1 - ix_2,\ x_0 + x_3)\Big) + w_A x_1,\ A = 1,2.
\end{aligned}$$

Here $w_1,\ w_2,\ w_3$ are arbitrary smooth real-valued functions of $x_0 + x_3$; U is an arbitrary function analytic in the first variable.

Substitution of Ansätze (6.1.26), (6.1.29) into equation (6.1.25) gives rise to the following systems of ODEs:

$$\begin{aligned}
1)\quad & i\gamma_1\dot{\varphi} = R,\\
2)\quad & i(\gamma_2 - \omega\gamma_1)\dot{\varphi} = R,\\
3)\quad & i(\gamma_0+\gamma_3)\dot{\varphi} + (\gamma_0+\gamma_3)(w_2\gamma_1\gamma_2 - w_1)\varphi = R,
\end{aligned}$$

where $R = (\bar{\varphi}\varphi)^{1/2k}\Big[g_1\Big(\bar{\varphi}\varphi(\bar{\varphi}\gamma_4\varphi)^{-1}\Big) + \gamma_4 g_2\Big(\bar{\varphi}\varphi(\bar{\varphi}\gamma_4\varphi)^{-1}\Big)\Big]\varphi.$

Generally speaking, Ansätze (6.1.26), (6.1.29) are not invariant with respect to the three-parameter subgroups of the group $\tilde{P}(1,3)$ (description of inequivalent $\tilde{P}(1,3)$-invariant Ansätze for the spinor field is given in Section 2.2). In the case involved we deal with reduction via conditionally-invariant Ansätze. For example, the involutive set of operators Q_a corresponding to the Ansatz 1 from (6.1.29) is of the form

$$\begin{aligned}
&Q_1 = (1/2)(\partial_0 - \partial_3), \quad Q_2 = w_1\partial_2 + (1/2)(1-2k)\dot{w}_1\gamma_2(\gamma_0+\gamma_3),\\
&Q_3 = (1/2)w_1(\partial_0+\partial_3) - \dot{w}_1 x_A\partial_A - \dot{w}_2\partial_1 - k\dot{w}_1 + (2w_1)^{-1}\\
&\quad\times(2\dot{w}_1^2 - w_1\ddot{w}_1)\Big(\gamma_A x_A + 2(k-1)\gamma_2 x_2\Big)(\gamma_0+\gamma_3) + (2w_1)^{-1}\\
&\quad\times\Big((2\dot{w}_1\dot{w}_2 - w_1\ddot{w}_2)\gamma_1 + (w_3\dot{w}_1 - w_1\dot{w}_3)\gamma_2\Big)(\gamma_0+\gamma_3).
\end{aligned}$$

The above operators satisfy the following relations:

$$\begin{aligned}
&[Q_1,\, Q_2] = [Q_1,\, Q_3] = 0, \quad [Q_2,\, Q_3] = -2w_1 Q_2,\\
&\tilde{Q}_1 L = 0, \quad \tilde{Q}_2 L = A_1 L + A_2 Q_1\psi + A_3 Q_2\psi,\\
&\tilde{Q}_3 L = B_0 L + B_a Q_a\psi,
\end{aligned}$$

where $\tilde{Q}_a$ is the first prolongation of operator Q_a; $L = i\gamma_\mu\partial_\mu\psi - (\bar{\psi}\psi)^{1/2k}(g_1 + g_2\gamma_4)\psi$; $A_a,\ B_0,\ B_a$ are some variable (4×4)-matrices. Hence it follows that the nonlinear Dirac equation (6.1.25) is conditionally-invariant with respect to the involutive set of operators $Q_1,\ Q_2,\ Q_3$.

In conclusion we adduce the two classes of new exact solutions of the nonlinear spinor equation

$$\Big(i\gamma_\mu\partial_\mu - \lambda(\bar{\psi}\psi)^{1/2k}\Big)\psi = 0$$

constructed with the use of conditionally-invariant Ansätze (6.1.10), (6.1.29)

$$\begin{aligned}
\psi(x) \quad = \quad & \exp\Big\{\gamma_1(\gamma_0+\gamma_3)\Big[(x_1\sin w_2 - x_2\cos w_2)\Big[\Big((1/2)\dot{w}_1 \\
& +Ce^{-w_1}\Big)\sin w_2 - (1/2)\dot{w}_2\cos w_2\Big] + w_4\sin w_2 + (1/2)\dot{w}_3 \\
& \times\cos w_2\Big] - \gamma_2(\gamma_0+\gamma_3)\Big[(x_1\sin w_2 - x_2\cos w_2)\Big[\Big((1/2)\dot{w}_1 \\
& +Ce^{-w_1}\Big)\cos w_2 + (1/2)\dot{w}_2\sin w_2\Big] + w_4\cos w_2 \\
& -(1/2)\dot{w}_3\sin w_2\Big]\Big\}\exp\{(1/2)w_1\gamma_0\gamma_3 + (1/2)w_2\gamma_1\gamma_2\} \\
& \times\exp\Big\{\Big(i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1 - C\gamma_1(\gamma_0+\gamma_3)\Big)(x_1\cos w_2 \\
& +x_2\sin w_2 + w_3)\Big\}\chi, \\
\psi(x) \quad = \quad & w_1^k\exp\Big\{(2w_1)^{-1}\Big[(\dot{w}_1x_1 + \dot{w}_2)\gamma_1 + \Big((2k-1)\dot{w}_1x_1 + w_3\Big)\gamma_2\Big] \\
& \times(\gamma_0+\gamma_3)\Big\}\exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}(w_1x_1 + w_2)\}\chi.
\end{aligned}$$

Here w_1, w_2, w_3, w_4 are arbitrary smooth real-valued functions of x_0+x_3; χ is an arbitrary four-component constant column.

6.2. Non-Lie reduction of Galilei-invariant spinor equations

Taking into account the classical ideas and methods of symmetry analysis of differential equations we generalize results obtained in the previous section in the form of the following non-Lie algorithm of reduction of PDEs:

- the maximal (in Lie sense) invariance group of the equation under study is found by the Lie method;

- subgroup analysis of the invariance group is carried out, each subgroup giving rise to some Ansatz which reduces PDE in question to an equation having a smaller dimension. As a rule, Ansätze obtained in this way have a quite definite structure which is determined by the representation of the symmetry group.
- the general form of the invariant Ansatz is obtained. This Ansatz includes several scalar functions $\theta_1, \ldots, \theta_N$ satisfying some compatible over-determined system of nonlinear PDEs (reduction conditions).
- equations for $\theta_1, \ldots, \theta_N$ are integrated.

Let us realize the above algorithm for the following system of nonlinear spinor PDEs:

$$\{-i(\gamma_0 + \gamma_4)\partial_t + i\gamma_a\partial_a + m(\gamma_0 - \gamma_4) - F(\psi^*, \psi)\}\psi = 0, \tag{6.2.1}$$

where F is a variable (4×4)-matrix.

According to Theorem 4.1.5 equation (6.2.1) is invariant under the Galilei group iff

$$F = \tilde{f}_1(\bar{\psi}\psi,\ \psi^\dagger\psi + \bar{\psi}\gamma_4\psi) + \tilde{f}_2(\bar{\psi}\psi,\ \psi^\dagger\psi + \bar{\psi}\gamma_4\psi)(\gamma_0 + \gamma_4), \tag{6.2.2}$$

where $\{\tilde{f}_1, \tilde{f}_2\} \subset C^1(\mathbb{R}^2, \mathbb{C}^2)$ are arbitrary functions. In Section 4.2 we have constructed $G(1,3)$-inequivalent Ansätze for the spinor field $\psi(t, \vec{x})$ invariant under three-parameter subgroups of the Galilei group. One can become convinced of the fact that these Ansätze have the form

$$\psi(t, \vec{x}) = \exp\{i\theta_0 + \gamma_a\theta_a(\gamma_0 + \gamma_4)\}\exp\{\theta_4\gamma_1\gamma_2\}\varphi(\omega), \tag{6.2.3}$$

where $\theta_\mu,\ \theta_4,\ \omega$ are smooth real-valued functions on $t,\ \vec{x}$; $\varphi = \varphi(\omega)$ is an arbitrary complex-valued four-component function.

In the following, we will describe all Ansätze (6.2.3) with $\theta_4 = 0$ reducing the Galilei-invariant equation (6.2.1), (6.2.2) to systems of ODEs.

Substituting (6.2.3) with $\theta_4 = 0$ into (6.2.1), (6.2.2) and requiring for the obtained equation be equivalent to a system of ordinary differential equations for $\varphi(\omega)$ we have

$$\begin{aligned}
&1)\quad \partial_2\theta_3 - \partial_3\theta_2 = f_1(\omega),\\
&2)\quad \partial_3\theta_1 - \partial_1\theta_3 = f_2(\omega),\\
&3)\quad \partial_1\theta_2 - \partial_2\theta_1 = f_3(\omega),
\end{aligned}$$

$$
\begin{aligned}
&4)\quad \partial_a\theta_a = f_4(\omega),\\
&5)\quad (\partial_t + 2\theta_a\partial_a)\theta_0 + 4m\theta_a\theta_a = f_5(\omega),\\
&6)\quad (\partial_t + 2\theta_a\partial_a)\omega = f_6(\omega),\\
&7)\quad \partial_a\omega = f_{6+a}(\omega),\\
&8)\quad \partial_a\theta_0 + 4m\theta_a = f_{9+a}(\omega).
\end{aligned} \tag{6.2.4}
$$

Here $f_1, \ldots, f_{12}$ are arbitrary smooth real-valued functions, $a = 1, 2, 3$.

As earlier (see Section 6.1), we introduce an equivalence relation E on the set of solutions of system of PDEs (6.2.4). We say that solutions of equations (6.2.4) $\theta_0(t,\vec{x}),\ \theta_a(t,\vec{x}),\ \omega(t,\vec{x})$ and $\theta_0'(t,\vec{x}),\ \theta_a'(t,\vec{x}),\ \omega'(t,\vec{x})$ are equivalent if they are transformed one into another by

1) a suitable transformation from the group $\widetilde{G}_{11}$ which is induced in the space of the variables $t,\ \vec{x},\ \theta_0(t,\vec{x}),\ \theta_a(t,\vec{x}),\ \omega(t,\vec{x})$ by the action of the Galilei group $G(1,3)$ on Ansatz (6.2.3), or

2) a suitable transformation of the form

$$
\begin{aligned}
&\theta_0 \to \theta_0 + h_0(\omega),\\
&\theta_a \to \theta_a + h_a(\omega),\\
&\omega \to h(\omega),
\end{aligned}
$$

where $\{h_\mu, h\} \subset C^1(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Theorem 6.2.1. *General solution of system of PDEs (6.2.4) determined up to the equivalence relation E is given by one of the following formulae:*

I. $m = 0$

1) $\omega = x_1 + w_1(t), \quad \theta_0 = C_3\Big(x_2 - 2w_2(t)\Big) + C_4\Big(x_3 - 2w_3(t)\Big) + C_5 t,$
$\theta_1 = -(1/2)\dot{w}_1(t), \quad \theta_2 = -\alpha(C_3x_2 + C_4x_3) + \dot{w}_2(t) + C_1x_2,$
$\theta_3 = \alpha(C_3x_3 - C_4x_2) + \dot{w}_3(t) + C_2x_2, \quad \alpha = (C_1C_3 + C_2C_4)$
$\times(C_3^2 + C_4^2)^{-1};$

2) $\omega = x_1 + w_0(t), \quad \theta_0 = C_3 t, \quad \theta_1 = -(1/2)\dot{w}_0(t),$
$\theta_2 = U(x_2 + ix_3, t) + U(x_2 - ix_3, t) + C_1x_2,$
$\theta_3 = iU(x_2 + ix_3, t) - iU(x_2 - ix_3, t) + C_2x_2;$

3) $\omega = t, \quad \theta_0 = x_a g_a(t), \quad \theta_a = \varepsilon_{abc}h_b(t)x_c + \partial_a W + w_0(t)x_a,$
function $W = W(t,\vec{x})$ being given by one of the relations $a - c$

a) *under* $g_1 = g_2 = g_3 = 0$
$\partial_a\partial_a W = 0;$

b) *under* $g_2 = 0,\ g_3 \neq 0$

$$W = g_3^{-1}\Big(r_1x_1x_3 + r_2x_2x_3 + r_4x_3 + (1/2)r_3x_3^2 - (1/2)g_3^{-1}g_1r_1x_3^2 + (1/2)(g_3^{-1}g_1r_1 - r_3)x_2^2\Big) + U(z,t) + U(z^*,t),$$

$$z = (g_1^2 + g_3^2)^{-1/2}(g_1x_3 - g_3x_1) + ix_2;$$

c) *under* $g_1^2 + g_2^2 \neq 0,\ g_3 \neq 0$

$$W = (1/2)g_3^{-2}r_1(2g_3x_1x_3 - g_1x_3^2) + (1/2)g_3^{-2}r_2(2g_3x_2x_3 - g_2x_3^2) + (1/2)g_3^{-1}(r_3x_3 + 2r_4x_3) + (1/2)g_3^2(g_1^2+g_2^2)^{-1}(r_1g_1 + r_2g_2 - r_3g_3)(g_2x_1 - g_1x_2)^2 + U(z,t) + U(z^*,t),$$

$$z = \Big((g_1^2+g_2^2)^{-1}(g_2^2+g_3^2) - g_1^2g_3^2(g_1^2+g_2^2)^{-2}\Big)^{1/2}(g_2x_1 - g_1x_2) + i\Big(g_1g_3(g_1^2+g_2^2)^{-1}(g_2x_1 - g_1x_2) + g_3x_2 - g_2x_3\Big),$$

where

$$r_a = -\Big(g_aw_0 + \varepsilon_{abc}g_bh_c + (1/2)\dot g_a\Big), \quad r_4 = g_0.$$

II. $m \neq 0$

1) $$\omega = x_1 + (4m)^{-1}C_5t^2 + C_7t, \quad \theta_0 = (2mC_7 + C_5t)\omega + (C_3 - 4mC_1)x_2 + (C_4 - 4mC_2)x_3 - (12m)^{-1}C_5^2t^3 - (1/2)C_5C_7t^2 + C_6t, \quad \theta_1 = -(4m)^{-1}C_5t - (1/2)C_7, \quad \theta_2 = C_1, \quad \theta_3 = C_2;$$

2) $$\omega = t, \quad \theta_0 = -2mR_0x_ax_a + R_ax_a - 4m(T_{ab}x_ax_b + T_ax_a), \quad \theta_a = R_0x_a + 2T_{ab}x_b + T_a,$$

where $R_0(t),\ R_b(t),\ T_{bc}(t),\ T_b(t)$ *are real-valued functions satisfying the Riccati-type systems of ODEs*

$$(\dot R_0 + 2R_0^2)\delta_{ab} + 2\dot T_{ab} + 8T_{ac}T_{bc} + 8R_0T_{ab} = 0,$$
$$\dot R_a - 4m\dot T_a - 8mR_0T_a - 16mT_{ab}T_b + 4T_{ab}R_b + 2R_0R_a = 0$$

and besides

$$T_{ab} = T_{ba}, \quad T_{11} + T_{22} + T_{33} = 0.$$

In the above formulae $w_0,\ w_a,\ g_0,\ g_a,\ h_a$ *are arbitrary smooth real-valued functions of* t*;* $a,b,c = 1,2,3$*;* U *is an arbitrary function analytic in the variable* z*;* $C_1,\ C_2,\ldots,C_7$ *are arbitrary constants.*

A detailed proof of this assumption can be found in [8, 303].

Substantial extension of the class of Ansätze (6.2.3) reducing nonlinear PDE (6.2.1), (6.2.2) to systems of ODEs as compared with the class of Lie

Ansätze (see Section 4.2) is achieved due to the conditional symmetry of equation (6.2.1).

Computing involutive sets of operators $Q_a = \xi_{a\mu}(t,\vec{x})\partial_\mu + \eta_a(t,\vec{x})$ ($\partial_0 \equiv \partial_t$), $a = 1, 2, 3$ with the use of formulae (6.1.20), (6.1.21), (4×4)-matrix $A = A(t,\vec{x})$ and scalar function $\omega(t,\vec{x})$ being determined by the formulae I.1–II.2, we can become convinced of that Ansätze I.2, I.3 correspond to conditional symmetry of system of PDEs (6.2.1), (6.2.2).

Substitution of the Ansätze obtained above into the initial equation (6.2.1), (6.2.2) yields systems of ODEs for a four-component function $\varphi(\omega)$

$$
\begin{aligned}
\text{I.}\quad & 1)\quad i\gamma_1\dot{\varphi} + i\Big((C_2\gamma_1 - C_1 - iC_5)(\gamma_0+\gamma_4) + iC_3\gamma_2 + iC_4\gamma_3\Big)\varphi = R,\\
& 2)\quad i\gamma_1\dot{\varphi} + i(C_2\gamma_1 - C_1 - iC_3)(\gamma_0+\gamma_4)\varphi = R,\\
& 3)\quad -i(\gamma_0+\gamma_4)\dot{\varphi} + i\Big((2h_a\gamma_a - 3w_0 - ig_0)(\gamma_0+\gamma_4)\\
& \qquad + ig_a\gamma_a\Big)\varphi = R,\\
\text{II.}\quad & 1)\quad i\gamma_1\dot{\varphi} + \Big((C_5\omega + C_6 - 4mC_1^2 - 4mC_2^2 + mC_7^2 + 2C_1C_3\\
& \qquad +2C_2C_4)(\gamma_0+\gamma_4) - C_3\gamma_2 - C_4\gamma_3 + m(\gamma_0-\gamma_4)\Big)\varphi = R,\\
& 2)\quad -i(\gamma_0+\gamma_4)\dot{\varphi} + \Big(-3iR_0(\gamma_0+\gamma_4) + (2R_aT_a - 4mT_aT_a)\\
& \qquad \times(\gamma_0+\gamma_4) - R_a\gamma_a + m(\gamma_0-\gamma_4)\Big)\varphi = R,
\end{aligned}
$$

Here $R = \Big(\tilde{f}_1(\bar{\varphi}\varphi, \varphi^\dagger\varphi + \bar{\varphi}\gamma_4\varphi) + \tilde{f}_2(\bar{\varphi}\varphi, \varphi^\dagger\varphi + \bar{\varphi}\gamma_4\varphi)(\gamma_0+\gamma_4)\Big)\varphi$; w_0, w_a, h_a, g_0, g_a, T_a, R_a are functions of ω determined in Theorem 6.2.1; $C_1, \ldots, C_7$ are constants.

A particular or general solution $\varphi = \varphi(\omega)$ of one of the above equations after being substituted into corresponding Ansatz (6.2.3) gives rise to a class of exact solutions of the initial nonlinear PDE.

As an example, we adduce the class of solutions of system of nonlinear PDEs (6.2.1), (6.2.2) with $m = 0$, $\tilde{f}_1 = 0$, $\tilde{f}_2 = \lambda(\psi^\dagger\psi + \bar{\psi}\gamma_4\psi)^k$, $\lambda = \mathrm{const}$, $k = \mathrm{const}$ constructed with use of the Ansatz I.3

$$\psi(x) = \exp\{i\lambda(\chi^\dagger\chi + \bar{\chi}\gamma_4\chi)^k t + (\gamma_0+\gamma_4)\gamma_a\partial_a W\}\chi,$$

where χ is an arbitrary constant four-component column, $W = W(t,\vec{x})$ is an arbitrary solution of the three-dimensional Laplace equation

$$\Delta_3 W = \partial_a\partial_a W = 0.$$

The approach to the problem of reduction of $G(1,3)$-invariant equations for the spinor field of the form (6.2.1) suggested above can be generalized for the case of an arbitrary Galilei-invariant system of PDEs admitting the group $G(1,3)$ with generators

$$\begin{aligned}
&P_0 = \partial_t, \quad P_a = \partial_a, \\
&J_{ab} = x_b\partial_a - x_a\partial_b + S_{ab}, \\
&G_a = t\partial_a + i\lambda x_a + \eta_a,
\end{aligned}$$

where $\lambda = \mathrm{const}$; $a, b = 1, 2, 3$; S_{ab}, η_a are arbitrary constant matrices satisfying the commutation relations of the Lie algebra $AE(3)$. Exact solutions of such a system are looked for in the form

$$\psi(t, \vec{x}) = \exp\{\theta_0 + \theta_a\eta_a\} \exp\{\theta_4 S_{12}\}\varphi(\omega),$$

where $\{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \omega\} \subset C^1(\mathbb{R}^4, \mathbb{R}^1)$.

REDUCTION AND EXACT SOLUTIONS OF $SU(2)$ YANG-MILLS EQUATIONS

In the present chapter a detailed account of symmetry properties of $SU(2)$ Yang-Mills equations is given. Using a subgroup structure of the Poincaré and conformal groups we have constructed all $C(1,3)$-inequivalent Ansätze for the Yang-Mills field which are invariant under three-parameter subgroups of the Poincaré group. With the aid of these Ansätze reduction of Yang-Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are obtained. A number of generalizations of the Lie Ansätze are suggested making it possible to construct broad families of exact solutions of the Yang-Mills equations containing arbitrary functions. It is shown that a possibility of such generalizations is provided by nontrivial conditional symmetry of the Yang-Mills equations.

7.1. Symmetry reduction and exact solutions of the Yang-Mills equations

1. Introduction. A majority of papers devoted to construction of the explicit form of exact solutions of the SU(2) Yang-Mills equations (YMEs)

$$\begin{gathered}\partial_\nu \partial^\nu \vec{A}_\mu - \partial^\mu \partial_\nu \vec{A}_\nu + e\Big((\partial_\nu \vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_\nu \vec{A}_\mu) \times \vec{A}_\nu \\ +(\partial^\mu \vec{A}_\nu) \times \vec{A}^\nu\Big) + e^2 \vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}.\end{gathered} \tag{7.1.1}$$

are based on the Ansätze for the three-component vector-potential of the Yang-Mills field $\vec{A}_\mu(x_0,\, x_1,\, x_2,\, x_3)$ (called, for brevity, the Yang-Mills field) suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek,

Witten (see [2] and references therein). And what is more, Ansätze mentioned are obtained in a non-algorithmic way, i.e., there is no regular and systematic method for constructing these Ansätze.

Since there are only a few distinct exact solutions of YMEs, it is difficult to give their reliable and self-consistent physical interpretation. That is why the problem of prime importance is the development of an effective regular approach for constructing new exact solutions of system of nonlinear PDEs (7.1.1).

A natural approach to construction of particular solutions of YMEs (7.1.1) is to utilize their symmetry properties. Apparatus of the theory of Lie transformation groups makes it possible to reduce system of PDEs (7.1.1) to systems of ODEs by using invariant Ansätze. If we succeed in constructing its general or particular solutions, then substituting the results obtained into the corresponding Ansätze we obtain exact solutions of YMEs. Let us note that symmetry reductions of the Euclidean self-dual YMEs (which form the first-order system of PDEs) by means of the subgroups of the Euclid group $E(4)$ have been performed in the paper [204]. It is interesting to note that many integrable two-dimensional PDEs are obtained as symmetry reductions of the self-dual YMEs (see [47] and references therein).

Another possibility of construction of exact solutions of YMEs is to use their conditional symmetry. To this end, we apply the same approach which enables us to obtain broad families of conditionally-invariant Ansätze for the nonlinear Dirac equation (see Chapter 6).

In the present chapter we exploit both possibilities mentioned above. In the first section symmetry reduction of system of PDEs (7.1.1) by means of three-parameter subgroups of the Poincaré group is carried out and a number of its non-Abelian exact solutions are constructed. The second section is devoted to investigation of conditional symmetry of YMEs.

2. Symmetry and solution generation for the Yang-Mills equations. It was known long ago that YMEs are invariant with respect to the group $C(1,3)\otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group having the following generators:

$$
\begin{aligned}
&P_\mu = \partial_\mu,\\
&J_{\alpha\beta} = x^\alpha\partial_\beta - x^\beta\partial_\alpha + A^{a\alpha}\partial_{A^a_\beta} - A^{a\beta}\partial_{A^a_\alpha},\\
&D = x_\mu\partial_\mu - A^a_\mu\partial_{A^a_\mu},\\
&K_\mu = 2x^\mu D - x_\nu x^\nu\partial_\mu + 2A^{a\mu}x_\nu\partial_{A^a_\nu} - 2A^a_\nu x^\nu\partial_{A^{a\mu}},
\end{aligned}
\tag{7.1.2}
$$

and $SU(2)$ is the infinite-parameter special unitary group with the following basis generator:

$$Q = \Big(\varepsilon_{abc} A^b_\mu w^c(x) + e^{-1}\partial_\mu w^a(x)\Big)\partial_{A^a_\mu}. \tag{7.1.3}$$

In (7.1.2), (7.1.3) $\partial_{A^a_\mu}=\partial/\partial A^a_\mu$, $w^c(x)$ are arbitrary smooth functions, ε_{abc} is the third-order anti-symmetric tensor with $\varepsilon_{123} = 1$.

But the fact that the group with generators (7.1.2), (7.1.3) is a maximal (in Lie sense) invariance group admitted by YMEs was established only recently [251] with the use of a symbolic computation technique. The only explanation for this situation is a very cumbersome structure of the system of PDEs (7.1.1). As a consequence, realization of the Lie algorithm of finding the maximal invariance group admitted by YMEs demands a huge amount of computations. This difficulty has been overcome with the aid of computer facilities.

One of the remarkable consequences of the fact that the equation under study admits a nontrivial symmetry group is a possibility of getting new solutions from the known ones by the solution generation technique (see Theorem 2.4.1).

To make use of Theorem 2.4.1 we need formulae for finite transformations generated by the infinitesimal operators (7.1.2), (7.1.3). We adduce them following [2, 137].

1) The group of translations (generator $X = \tau_\mu P_\mu$)

$$x'_\mu = x_\mu + \tau_\mu, \quad A^{d\prime}_\mu = A^d_\mu.$$

2) The Lorentz group $O(1, 3)$

a) the group of rotations (generator $X = \tau J_{ab}$)

$$\begin{aligned}
&x'_0 = 0, \quad x'_c = x_c, \quad c \neq a, \quad c \neq b,\\
&x'_a = x_a \cos\tau + x_b \sin\tau,\\
&x'_b = x_b \cos\tau - x_a \sin\tau,\\
&A^{d\prime}_0 = A^d_0, \quad A^{d\prime}_c = A^d_c, \quad c \neq a, \quad c \neq b,\\
&A^{d\prime}_a = A^d_a \cos\tau + A^d_b \sin\tau,\\
&A^{d\prime}_b = A^d_b \cos\tau - A^d_a \sin\tau;
\end{aligned}$$

b) the group of Lorentz transformations (generator $X = \tau J_{0a}$)

$$x'_0 = x_0 \cosh\tau + x_a \sinh\tau,$$

$$\begin{aligned}
&x'_a = x_a \cosh\tau + x_0 \sinh\tau, \quad x'_b = x_b, \quad b \neq a,\\
&A_0^{d\prime} = A_0^d \cosh\tau + A_a^d \sinh\tau,\\
&A_a^{d\prime} = A_a^d \cosh\tau + A_0^d \sinh\tau, \quad A_b^{d\prime} = A_b^d, \quad b \neq a.
\end{aligned}$$

3) The group of scale transformations (generator $X = \tau D$)

$$x'_\mu = x_\mu e^{\tau}, \quad A_\mu^{d\prime} = A_\mu^d e^{-\tau}.$$

4) The group of special conformal transformations (generator $X = \tau_\mu K^\mu$)

$$\begin{aligned}
&x'_\mu = (x_\mu - \tau_\mu x_\nu x^\nu)\sigma^{-1}(x),\\
&A_\mu^{d\prime} = \Big(g_{\mu\nu}\sigma(x) + 2(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha \tau_\mu x_\nu\\
&\quad -x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu\Big)A^{d\nu}.
\end{aligned}$$

5) The group of gauge transformations (generator $X = Q$)

$$\begin{aligned}
&x'_\mu = x_\mu,\\
&A_\mu^{d\prime} = A_\mu^d \cos w + \varepsilon_{dbc} A_\mu^b n^c \sin w + 2n^d n^b A_\mu^b \sin^2(w/2)\\
&\quad +e^{-1}\Big((1/2)n^d\partial_\mu w + (1/2)(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\Big).
\end{aligned}$$

In the above formulae $\sigma(x) = 1 - \tau_\alpha x^\alpha + (\tau_\alpha\tau^\alpha)(x_\beta x^\beta)$, $n^a = n^a(x)$ is the unit vector determined by the equality $w^a(x) = w(x)n^a(x)$, $a = 1, 2, 3$.

Using Theorem 2.4.1 it is not difficult to obtain formulae for generating solutions of YMEs by the above transformation groups. We adduce these omitting the derivation (see also [134]).

1) The group of translations

$$A_\mu^a(x) = u_\mu^a(x + \tau).$$

2) The Lorentz group

$$\begin{aligned}
A_\mu^d(x) = {}& a_\mu u_0^d(a\cdot x,\, b\cdot x,\, c\cdot x,\, d\cdot x) + b_\mu u_1^d(a\cdot x,\, b\cdot x,\, c\cdot x,\, d\cdot x)\\
&+c_\mu u_2^d(a\cdot x,\, b\cdot x,\, c\cdot x,\, d\cdot x) + d_\mu u_3^d(a\cdot x,\, b\cdot x,\, c\cdot x,\, d\cdot x).
\end{aligned}$$

3) The group of scale transformations

$$A_\mu^d(x) = e^{\tau} u_\mu^d(xe^{\tau}).$$

4) The group of special conformal transformations

$$A^d_\mu(x) = \Big(g_{\mu\nu}\sigma^{-1}(x) + 2\sigma^{-2}(x)(x_\nu\tau_\mu - x_\mu\tau_\nu + 2\tau_\alpha x^\alpha x_\mu\tau_\nu \\ -x_\alpha x^\alpha\tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu)\Big)u^{d\nu}\Big([x - \tau(x_\alpha x^\alpha)]\sigma^{-1}(x)\Big).$$

5) The group of gauge transformations

$$A^d_\mu(x) = u^d_\mu \cos w + \varepsilon_{dbc}u^b_\mu n^c \sin w + 2n^d n^b u^b_\mu \sin^2(w/2) \\ +e^{-1}\Big((1/2)n^d\partial_\mu w + (1/2)(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\Big).$$

Here $u^d_\mu(x)$ is a given solution of YMEs; $A^d_\mu(x)$ is a new solution of YMEs; $\tau,\ \tau_\mu$ are arbitrary parameters; $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ are arbitrary parameters satisfying the equalities

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

In addition, we use the following notations: $x+\tau = \{x_\mu+\tau_\mu, \mu = 0,\dots,3\}$, $a\cdot x = a_\mu x^\mu$.

Thus, each particular solution of YMEs gives rise to a multi-parameter family of exact solutions by virtue of the above solution generation formulae.

3. Ansätze for the Yang-Mills field. Let us recall that the key idea of the symmetry approach to the problem of reduction of PDEs is a special choice of the form of a solution. This choice is dictated by a structure of the symmetry group admitted by the equation under study.

In the case involved, to reduce YMEs by N variables we have to construct Ansätze for the Yang-Mills field $A^a_\mu(x)$ invariant under $(4-N)$-dimensional subalgebras of the algebra with the basis elements (7.1.2), (7.1.3). Since we are looking for Poincaré-invariant Ansätze reducing YMEs to systems of ODEs, N is equal to 3. Due to invariance of YMEs under the conformal group $C(1,3)$ it is enough to consider only subalgebras which cannot be transformed one into another by a group transformation from $C(1,3)$, i.e., $C(1,3)$-inequivalent subalgebras. Complete description of $C(1,3)$-inequivalent subalgebras of the Poincaré algebra was obtained in [100].

According to Theorem 1.5.1 to construct an Ansatz invariant under the invariance algebra having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \eta^b_{a\mu}(x,A)\partial_{A^b_\mu},\quad a = 1,2,3, \tag{7.1.4}$$

where $A = \{A^a_\mu,\ a = 1,2,3,\ \mu = 0,\dots,3\}$, we have

- to construct a complete system of functionally-independent invariants of the operators (7.1.4) $\Omega = \{\omega_j(x, A),\ j = 1, \dots, 13\}$;
- to resolve the relations

$$F_j\Big(\omega_1(x, A), \dots, \omega_{13}(x, A)\Big) = 0, \quad j = 1, \dots, 12 \tag{7.1.5}$$

with respect to the functions A^a_μ.

As a result, we get an Ansatz for the field $A^a_\mu(x)$ which reduces YMEs to the system of twelve nonlinear ODEs.

Remark 7.1.1. Equalities (7.1.5) can be resolved with respect to $A^a_\mu,\ a = 1, 2, 3,\ \mu = 0, \dots, 3$ provided the condition

$$\mathrm{rank}\, \|\xi_{a\mu}(x, A)\|^{3\ \ 3}_{a=1 \mu=0} = 3 \tag{7.1.6}$$

holds. If (7.1.6) does not hold, the above procedure leads to partially-invariant solutions [235], which are not considered here.

In Section 1.5 we have established that a procedure of construction of invariant Ansätze could be substantially simplified if coefficients of operators X_a have the structure:

$$\xi_{a\mu} = \xi_{a\mu}(x), \quad \eta^b_{a\mu} = \rho^{bc}_{a\mu\nu}(x) A^c_\nu \tag{7.1.7}$$

(i.e., basis elements of the invariance algebra realize a linear representation). In this case, the invariant Ansatz for the field $A^a_\mu(x)$ is searched for in the form

$$A^a_\mu(x) = Q^{ab}_{\mu\nu}(x) B^{b\nu}\Big(\omega(x)\Big). \tag{7.1.8}$$

Here $B^b_\nu(\omega)$ are arbitrary smooth functions and $\omega(x),\ Q^{ab}_{\mu\nu}(x)$ are particular solutions of the system of PDEs

$$\xi_{a\mu}\omega_{x_\mu} = 0, \quad (\xi_{a\nu}\partial_\nu - \rho^{bc}_{a\mu\alpha}) Q^{cd}_{\alpha\beta} = 0, \tag{7.1.9}$$

where $\mu = 0, \dots, 3,\ a, b, d = 1, 2, 3$.

The basis elements of the Poincaré algebra $P_\mu,\ J_{\alpha\beta}$ from (7.1.2) evidently satisfy conditions (7.1.7) and besides the equalities

$$\eta^b_{a\mu} = \rho_{a\mu\nu}(x) A^b_\nu, \tag{7.1.10}$$

hold.

This fact allows further simplification of formulae (7.1.8), (7.1.9). Namely, the Ansatz for the Yang-Mills field invariant under a 3-dimensional subalgebra of the Poincaré algebra with basis elements belonging to the class (7.1.4), (7.1.10) should be looked for in the form

$$A_\mu^a(x) = Q_{\mu\nu}(x) B^{a\nu}\Big(\omega(x)\Big), \tag{7.1.11}$$

where $B_\nu^a(\omega)$ are arbitrary smooth functions and $\omega(x)$, $Q_{\mu\nu}(x)$ are particular solutions of the following system of PDEs:

$$\xi_{a\mu}\omega_{x_\mu} = 0, \quad a = 1, 2, 3, \tag{7.1.12}$$
$$\xi_{a\alpha}\partial_\alpha Q_{\mu\nu} - \rho_{a\mu\alpha}Q_{\alpha\nu} = 0, \quad a = 1, 2, 3, \quad \mu, \nu = 0, \ldots, 3. \tag{7.1.13}$$

Thus, to obtain the complete description of $C(1,3)$-inequivalent Ansätze for the field $A_\mu^a(x)$ invariant under 3-dimensional subalgebras of the Poincaré algebra, it is necessary to integrate the over-determined system of PDEs (7.1.12), (7.1.13) for each $C(1,3)$-inequivalent subalgebra. Let us note that compatibility of (7.1.12), (7.1.13) is guaranteed by the fact that operators $X_1,\ X_2,\ X_3$ form a Lie algebra.

Consider, as an example, a procedure of constructing Ansatz (7.1.11) invariant under the subalgebra $\langle P_1,\ P_2,\ J_{03}\rangle$. In this case system (7.1.12) reads

$$\omega_{x_1} = 0, \quad \omega_{x_2} = 0, \quad x_0\omega_{x_3} + x_3\omega_{x_0} = 0,$$

whence $\omega = x_0^2 - x_3^2$.

Next, we note that the coefficients $\rho_{1\mu\nu},\ \rho_{2\mu\nu}$ of the operators $P_1,\ P_2$ are equal to zero, while coefficients $\rho_{3\mu\nu}$ form the following (4×4) matrix

$$\|\rho_{3\mu\nu}\|_{\mu,\nu=0}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(we designate this constant matrix by the symbol S).

With account of the above fact, equations (7.1.13) take the form

$$Q_{x_1} = 0, \quad Q_{x_2} = 0, \quad x_0 Q_{x_3} + x_3 Q_{x_0} - SQ = 0, \tag{7.1.14}$$

where $Q = \|Q_{\mu\nu}(x)\|_{\mu,\nu=0}^3$ is a (4×4)-matrix.

From the first two equations of system (7.1.14) it follows that $Q = Q(x_0, x_3)$. Since S is a constant matrix, a solution of the third equation can be looked for in the form (see Section 2.2)

$$Q = \exp\{f(x_0,\, x_3)S\}.$$

Substituting this expression into (7.1.14) we get

$$(x_0 f_{x_3} + x_3 f_{x_0} - 1)\exp\{fS\} = 0$$

or, equivalently,

$$x_0 f_{x_3} + x_3 f_{x_0} = 1,$$

whence $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (7.1.14) reads

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

Using an evident identity $S = S^3$ we get the equalities:

$$\begin{aligned} Q \quad &= \quad \sum_{n=0}^{\infty} \frac{S^n}{n!}\Big(\ln(x_0+x_3)\Big)^n = I + S\bigg(\ln(x_0+x_3) + \frac{1}{3!}[\ln(x_0+x_3)]^3 \\ &\quad + \ldots\bigg) + S^2\bigg(\frac{1}{2!}[\ln(x_0+x_3)]^2 + \frac{1}{4!}[\ln(x_0+x_3)]^4 + \ldots\bigg) \\ &\quad = I + S\sinh[\ln(x_0+x_3)] + S^2\Big(\cosh[\ln(x_0+x_3)] - 1\Big), \end{aligned}$$

where I is the unit (4×4)-matrix.

Substitution of the obtained expressions for functions $\omega(x)$, $Q_{\mu\nu}(x)$ into (7.1.11) yields the Ansatz for the Yang-Mills field $A^a_\mu(x)$ invariant under the algebra $\langle P_1,\, P_2,\, J_{03}\rangle$

$$\begin{aligned} A^a_0 &= B^a_0(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_3(x_0^2 - x_3^2)\sinh\ln(x_0+x_3), \\ A^a_1 &= B^a_1(x_0^2 - x_3^2), \quad A^a_2 = B^a_2(x_0^2 - x_3^2), \qquad\qquad (7.1.15) \\ A^a_3 &= B^a_3(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_0(x_0^2 - x_3^2)\sinh\ln(x_0+x_3). \end{aligned}$$

Substituting (7.1.15) into YMEs we get a system of ODEs for functions B^a_μ. If we succeed in constructing its general or particular solution, then substituting it into formulae (7.1.15) we get an exact solution of YMEs. But such a solution will have an unpleasant feature: independent variables x_μ will be included into it in an asymmetric way. At the same time, in the initial equation

(7.1.1) all independent variables are on equal rights. To remove this drawback we have to apply the solution generation procedure by transformations from the Lorentz group. As a result, we will obtain the Ansatz for the Yang-Mills field in the manifestly-covariant form with symmetric dependence on x_μ.

In the same way, we construct the rest of Ansätze invariant under three-dimensional subalgebras of the Poincaré algebra. They are represented in the unified form (7.1.11), where

$$\begin{aligned} Q_{\mu\nu}(x) \;=\; & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 \\ & +2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 \\ & -\theta_1\sin\theta_3)c_\nu + (\theta_1^2+\theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu \\ & -b_\nu c_\mu)\sin\theta_3 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0} \\ & \times(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu) \end{aligned} \tag{7.1.16}$$

and $\theta_\mu(x)$, $\omega(x)$ are some functions whose explicit form is determined by the choice of a subalgebra of the Poincaré algebra $AP(1,3)$.

Below, we adduce a complete list of 3-dimensional $C(1,3)$-inequivalent subalgebras of the Poincaré algebra following [100]

$$\begin{aligned} & L_1 = \langle P_0,\, P_1,\, P_2\rangle; \quad L_2 = \langle P_1,\, P_2,\, P_3\rangle; \\ & L_3 = \langle P_0+P_3,\, P_1,\, P_2\rangle; \quad L_4 = \langle J_{03}+\alpha J_{12},\, P_1,\, P_2\rangle; \\ & L_5 = \langle J_{03},\, P_0+P_3,\, P_1\rangle; \quad L_6 = \langle J_{03}+P_1,\, P_0,\, P_3\rangle; \\ & L_7 = \langle J_{03}+P_1,\, P_0+P_3,\, P_2\rangle; \quad L_8 = \langle J_{12}+\alpha J_{03},\, P_0,\, P_3\rangle; \\ & L_9 = \langle J_{12}+P_0,\, P_1,\, P_2\rangle; \quad L_{10} = \langle J_{12}+P_3,\, P_1,\, P_2\rangle; \\ & L_{11} = \langle J_{12}+P_0-P_3,\, P_1,\, P_2\rangle; \quad L_{12} = \langle G_1,\, P_0+P_3,\, P_2+\alpha P_1\rangle; \\ & L_{13} = \langle G_1+P_2,\, P_0+P_3,\, P_1\rangle; \quad L_{14} = \langle G_1+P_0-P_3,\, P_0+P_3,\, P_2\rangle; \\ & L_{15} = \langle G_1+P_0-P_3,\, P_0+P_3,\, P_1+\alpha P_2\rangle; \quad L_{16} = \langle J_{12},\, J_{03},\, P_0+P_3\rangle; \\ & L_{17} = \langle G_1+P_2,\, G_2-P_1+\alpha P_2,\, P_0+P_3\rangle; \quad L_{18} = \langle J_{03},\, G_1,\, P_2\rangle; \\ & L_{19} = \langle G_1,\, J_{03},\, P_0+P_3\rangle; \quad L_{20} = \langle G_1,\, J_{03}+P_2,\, P_0+P_3\rangle; \\ & L_{21} = \langle G_1,\, J_{03}+P_1+\alpha P_2,\, P_0+P_3\rangle; \quad L_{22} = \langle G_1,\, G_2,\, J_{03}+\alpha J_{12}\rangle; \\ & L_{23} = \langle G_1,\, P_0+P_3,\, P_1\rangle; \quad L_{24} = \langle J_{12},\, P_1,\, P_2\rangle; \\ & L_{25} = \langle J_{03},\, P_0,\, P_3\rangle; \quad L_{26} = \langle J_{12},\, J_{13},\, J_{23}\rangle; \quad L_{27} = \langle J_{01},\, J_{02},\, J_{12}\rangle. \end{aligned} \tag{7.1.17}$$

Here $G_i = J_{0i} - J_{i3},\ i = 1, 2,\ \alpha \in \mathbb{R}^1$.

$P(1,3)$-invariant Ansätze for the Yang-Mills field $A_\mu^a(x)$ are of the form (7.1.11), (7.1.16), functions $\theta_\mu(x)$, $\omega(x)$ being determined by one of the fol-

lowing formulae:

$$
\begin{aligned}
L_1 &: \theta_\mu = 0, \quad \omega = d\cdot x;\\
L_2 &: \theta_\mu = 0, \quad \omega = a\cdot x;\\
L_3 &: \theta_\mu = 0, \quad \omega = k\cdot x;\\
L_4 &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1=\theta_2=0, \quad \theta_3 = \alpha\ln|k\cdot x|,\\
&\quad\ \omega = (a\cdot x)^2 - (d\cdot x)^2;\\
L_5 &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1=\theta_2=\theta_3=0, \quad \omega = c\cdot x;\\
L_6 &: \theta_0 = -b\cdot x, \quad \theta_1=\theta_2=\theta_3=0, \quad \omega = c\cdot x;\\
L_7 &: \theta_0 = -b\cdot x, \quad \theta_1=\theta_2=\theta_3=0, \quad \omega = b\cdot x - \ln|k\cdot x|;\\
L_8 &: \theta_0 = \alpha\arctan(b\cdot x/c\cdot x), \quad \theta_1=\theta_2=0, \quad \theta_3 = -\arctan(b\cdot x/c\cdot x),\\
&\quad\ \omega = (b\cdot x)^2 + (c\cdot x)^2;\\
L_9 &: \theta_0=\theta_1=\theta_2=0, \quad \theta_3 = -a\cdot x, \quad \omega = d\cdot x;\\
L_{10} &: \theta_0=\theta_1=\theta_2=0, \quad \theta_3 = d\cdot x, \quad \omega = a\cdot x;\\
L_{11} &: \theta_0=\theta_1=\theta_2=0, \quad \theta_3 = -(1/2)k\cdot x, \quad \omega = a\cdot x - d\cdot x;\\
L_{12} &: \theta_0 = 0, \quad \theta_1 = (1/2)(b\cdot x - \alpha c\cdot x)(k\cdot x)^{-1}, \quad \theta_2=\theta_3=0,\\
&\quad\ \omega = k\cdot x; \qquad\qquad (7.1.18)\\
L_{13} &: \theta_0=\theta_2=\theta_3=0, \quad \theta_1 = (1/2)c\cdot x, \quad \omega = k\cdot x;\\
L_{14} &: \theta_0=\theta_2=\theta_3=0, \quad \theta_1 = -(1/4)k\cdot x, \quad \omega = 4b\cdot x + (k\cdot x)^2;\\
L_{15} &: \theta_0=\theta_2=\theta_3=0, \quad \theta_1 = -(1/4)k\cdot x, \quad \omega = 4(\alpha b\cdot x - c\cdot x)\\
&\quad\ +\alpha(k\cdot x)^2;\\
L_{16} &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1=\theta_2=0, \quad \theta_3 = -\arctan(b\cdot x/c\cdot x),\\
&\quad\ \omega = (b\cdot x)^2 + (c\cdot x)^2;\\
L_{17} &: \theta_0=\theta_3=0, \quad \theta_1 = (1/2)\Big(c\cdot x + (\alpha + k\cdot x)b\cdot x\Big)\Big(1 + k\cdot x\\
&\quad\ \times(\alpha + k\cdot x)\Big)^{-1}, \quad \theta_2 = -(1/2)(b\cdot x - c\cdot xk\cdot x)\Big(1 + k\cdot x\\
&\quad\ \times(\alpha + k\cdot x)\Big)^{-1}, \quad \omega = k\cdot x;\\
L_{18} &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1 = (1/2)b\cdot x(k\cdot x)^{-1}, \quad \theta_2=\theta_3=0,\\
&\quad\ \omega = (a\cdot x)^2 - (b\cdot x)^2 - (d\cdot x)^2;\\
L_{19} &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1 = (1/2)b\cdot x(k\cdot x)^{-1}, \quad \theta_2=\theta_3=0, \quad \omega = c\cdot x;\\
L_{20} &: \theta_0 = -\ln|k\cdot x|, \quad \theta_1 = (1/2)b\cdot x(k\cdot x)^{-1}, \quad \theta_2=\theta_3=0,\\
&\quad\ \omega = \ln|k\cdot x| - c\cdot x;
\end{aligned}
$$

$$
\begin{array}{lll}
L_{21} & : & \theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \dfrac{1}{2}(b\cdot x - \ln|k\cdot x|)(k\cdot x)^{-1}, \quad \theta_2 = \theta_3 = 0, \\
 & & \omega = \alpha \ln|k\cdot x| - c\cdot x; \\
L_{22} & : & \theta_0 = -\ln|k\cdot x|, \quad \theta_1 = (1/2)b\cdot x(k\cdot x)^{-1}, \quad \theta_2 = (1/2)c\cdot x(k\cdot x)^{-1}, \\
 & & \theta_3 = \alpha\ln|k\cdot x|, \quad \omega = (a\cdot x)^2 - (b\cdot x)^2 - (c\cdot x)^2 - (d\cdot x)^2.
\end{array}
$$

Here $k_\mu = a_\mu + d_\mu,\ \mu = 0, \ldots, 3.$

Note 7.1.2. Basis elements of subalgebras $L_{23} - L_{27}$ do not satisfy (7.1.6). That is why Ansätze invariant under these subalgebras lead to partially-invariant solutions and are not considered here.

4. Reduction of the Yang-Mills equations. In order to reduce YMEs to ODE it is necessary to substitute Ansatz (7.1.11), (7.1.16) into (7.1.1) and convolute the expression obtained with $Q^\mu_\alpha(x)$. As a result, we get a system of twelve nonlinear ODEs for functions $B^a_\nu(\omega)$ of the form

$$
\begin{array}{l}
k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eg_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma \\
\quad + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}.
\end{array}
\tag{7.1.19}
$$

Coefficients of the reduced ODE are given by the following formulae:

$$
\begin{array}{rcl}
k_{\mu\gamma} & = & g_{\mu\gamma}F_1 - G_\mu G_\gamma, \\
l_{\mu\gamma} & = & g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\
m_{\mu\gamma} & = & R_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\
g_{\mu\nu\gamma} & = & g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\
h_{\mu\nu\gamma} & = & (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma},
\end{array}
\tag{7.1.20}
$$

where $g_{\mu\nu}$ is a metric tensor of the Minkowski space $R(1,3)$ and $F_1,\ F_2,\ G_\mu$, $\ldots$, $T_{\mu\nu\gamma}$ are functions of ω determined by the relations

$$
\begin{array}{l}
F_1 = \omega_{x_\mu}\omega_{x^\mu}, \quad F_2 = \Box\omega, \quad G_\mu = Q_{\alpha\mu}\omega_{x_\alpha}, \quad H_\mu = Q_{\alpha\mu x_\alpha}, \\
S_{\mu\nu} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}\omega_{x^\beta}, \quad R_{\mu\nu} = Q^\alpha_\mu \Box Q_{\alpha\nu}, \\
T_{\mu\nu\gamma} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}Q_{\beta\gamma} + Q^\alpha_\nu Q_{\alpha\gamma x_\beta}Q_{\beta\mu} + Q^\alpha_\gamma Q_{\alpha\mu x_\beta}Q_{\beta\nu}.
\end{array}
\tag{7.1.21}
$$

Substituting functions $Q_{\mu\nu}(x)$ from (7.1.16), where $\theta_\mu(x),\ \omega(x)$ are determined by one of the formulae (7.1.18), into (7.1.20), (7.1.21) we obtain coefficients of the corresponding systems of ODEs (7.1.19)

$$
L_1 \quad : \quad k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,
$$

$$
\begin{aligned}
& && g_{\mu\nu\gamma} = g_{\mu\gamma} d_\nu + g_{\nu\gamma} d_\mu - 2g_{\mu\nu} d_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
& L_2 &:& \ k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} a_\nu + g_{\nu\gamma} a_\mu - 2g_{\mu\nu} a_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
& L_3 &:& \ k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
& L_4 &:& \ k_{\mu\gamma} = 4g_{\mu\gamma}\omega - a_\mu a_\gamma(\omega+1)^2 - d_\mu d_\gamma(\omega-1)^2 - (a_\mu d_\gamma + a_\gamma d_\mu) \\
& && \times(\omega^2-1), \quad l_{\mu\gamma} = 4\Big(g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma)\Big) - 2k_\mu(a_\gamma - d_\gamma + k_\gamma\omega), \\
& && m_{\mu\gamma} = 0, \quad g_{\mu\nu\gamma} = \epsilon\Big(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu\omega) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu\omega) \\
& && -2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma\omega)\Big), \quad h_{\mu\nu\gamma} = (\epsilon/2)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \alpha\epsilon\Big((b_\mu c_\nu \\
& && -c_\mu b_\nu)k_\gamma + (b_\nu c_\gamma - c_\nu b_\gamma)k_\mu + (b_\gamma c_\mu - c_\gamma b_\mu)k_\nu\Big); \\
& L_5 &:& \ k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = -\epsilon c_\mu k_\gamma, \quad m_{\mu\gamma} = 0, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \quad h_{\mu\nu\gamma} = (\epsilon/2)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
& L_6 &:& \ k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \quad h_{\mu\nu\gamma} = -\Big((a_\mu d_\nu - a_\nu d_\mu)b_\gamma \\
& && +(a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu\Big); \\
& L_7 &:& \ k_{\mu\gamma} = -g_{\mu\gamma} - (b_\mu - \epsilon k_\mu \mathrm{e}^\omega)(b_\gamma - \epsilon k_\gamma \mathrm{e}^\omega), \quad l_{\mu\gamma} = -2(a_\mu d_\gamma - a_\gamma d_\mu) \\
& && +\epsilon \mathrm{e}^\omega k_\gamma(b_\mu - \epsilon k_\mu \mathrm{e}^\omega), \quad m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma}(b_\nu - \epsilon k_\nu \mathrm{e}^\omega) + g_{\nu\gamma}(b_\mu - \epsilon k_\mu \mathrm{e}^\omega) - 2g_{\mu\nu}(b_\gamma - \epsilon k_\gamma \mathrm{e}^\omega), \\
& && h_{\mu\nu\gamma} = -\Big((a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu\Big); \\
& L_8 &:& \ k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma), \\
& && m_{\mu\gamma} = -\omega^{-1}\Big(\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) + b_\mu b_\gamma\Big), \quad g_{\mu\nu\gamma} = 2\omega^{1/2}(g_{\mu\gamma} c_\nu \\
& && +g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma), \quad h_{\mu\nu\gamma} = (1/2)\omega^{-1/2}(g_{\mu\gamma} c_\nu - g_{\mu\nu} c_\gamma) + \alpha\omega^{-1/2} \\
& && \times\Big((a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - d_\nu a_\gamma)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu\Big); \\
& L_9 &:& \ k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} d_\nu + g_{\nu\gamma} d_\mu - 2g_{\mu\nu} d_\gamma, \quad h_{\mu\nu\gamma} = a_\gamma(b_\mu c_\nu - c_\mu b_\nu) \\
& && +a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\gamma c_\mu - c_\gamma b_\mu); \\
& L_{10} &:& \ k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma), \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} a_\nu + g_{\nu\gamma} a_\mu - 2g_{\mu\nu} a_\gamma, \quad h_{\mu\nu\gamma} = -\Big(d_\gamma(b_\mu c_\nu - c_\mu b_\nu)
\end{aligned}
$$

$$
\begin{aligned}
& && +d_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + d_\nu(b_\gamma c_\mu - c_\gamma b_\mu)\Big); \\
L_{11} &: && k_{\mu\gamma} = -(a_\mu - d_\mu)(a_\gamma - d_\gamma), \quad l_{\mu\gamma} = -2(b_\mu c_\gamma - c_\mu b_\gamma), \quad m_{\mu\gamma} = 0, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma}(a_\nu - d_\nu) + g_{\nu\gamma}(a_\mu - d_\mu) - 2g_{\mu\nu}(a_\gamma - d_\gamma), \quad h_{\mu\nu\gamma} = \\
& && = (1/2)\Big(k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)\Big); \\
L_{12} &: && k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\omega^{-1} k_\mu k_\gamma, \quad m_{\mu\gamma} = -\alpha^2\omega^{-2} k_\mu k_\gamma, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma, \quad h_{\mu\nu\gamma} = (1/2)\omega^{-1}(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) \\
& && +\alpha\omega^{-1}\Big((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu\Big); \\
L_{13} &: && k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& && g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma, \quad h_{\mu\nu\gamma} = -\Big((k_\mu b_\nu - k_\nu b_\mu)c_\gamma \\
& && +(k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu\Big); \\
L_{14} &: && k_{\mu\gamma} = -16(g_{\mu\gamma} + b_\mu b_\gamma), \quad l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0, \qquad (7.1.22) \\
& && g_{\mu\nu\gamma} = 4(g_{\mu\gamma} b_\nu + g_{\nu\gamma} b_\mu - 2g_{\mu\nu} b_\gamma); \\
L_{15} &: && k_{\mu\gamma} = -16\Big((1+\alpha^2)g_{\mu\gamma} + (c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma)\Big), \\
& && l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0, \quad g_{\mu\nu\gamma} = -4\Big(g_{\mu\gamma}(c_\nu - \alpha b_\nu) \\
& && +g_{\nu\gamma}(c_\mu - \alpha b_\mu) - 2g_{\mu\nu}(c_\gamma - \alpha b_\gamma)\Big); \\
L_{16} &: && k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma) - 2\epsilon\omega^{1/2} k_\gamma c_\mu, \\
& && m_{\mu\gamma} = -\omega^{-1} b_\mu b_\gamma, \quad g_{\mu\nu\gamma} = 2\omega^{1/2}(g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma), \\
& && h_{\mu\nu\gamma} = (1/2)\Big(\epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) + \omega^{-1/2}(g_{\mu\gamma} c_\nu - g_{\mu\nu} c_\gamma)\Big); \\
L_{17} &: && k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -(2\omega + \alpha)\Big(\omega(\omega + \alpha) + 1\Big) k_\mu k_\gamma, \\
& && m_{\mu\gamma} = -4k_\mu k_\gamma\Big(1 + \omega(\omega + \alpha)\Big)^{-2}, \quad g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma, \\
& && h_{\mu\nu\gamma} = (1/2)(2\omega + \alpha)\Big(1 + \omega(\omega + \alpha)\Big)^{-1}(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) \\
& && -2\Big(1 + \omega(\omega + \alpha)\Big)^{-1}\Big((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu \\
& && +(k_\gamma b_\mu - k_\mu b_\gamma)c_\nu\Big); \\
L_{18} &: && k_{\mu\gamma} = 4\omega g_{\mu\gamma} - (k_\mu\omega + a_\mu - d_\mu)(k_\gamma\omega + a_\gamma - d_\gamma), \quad l_{\mu\gamma} = 6g_{\mu\gamma} \\
& && +4(a_\mu d_\gamma - a_\gamma d_\mu) - 3k_\gamma(k_\mu\omega + a_\mu - d_\mu), \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& && g_{\mu\nu\gamma} = \epsilon\Big(g_{\mu\gamma}(k_\nu\omega + a_\nu - d_\nu) + g_{\nu\gamma}(k_\mu\omega + a_\mu - d_\mu)
\end{aligned}
$$

$$-2g_{\mu\nu}(k_\gamma\omega + a_\gamma - d_\gamma)\Big), \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{19} \ : \ k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{20} \ : \ k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \epsilon k_\mu)(c_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu - 2k_\mu k_\gamma,$$
$$m_{\mu\gamma} = -k_\mu k_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}(\epsilon k_\nu - c_\nu) + g_{\nu\gamma}(\epsilon k_\mu - c_\mu)$$
$$-2g_{\mu\nu}(\epsilon k_\gamma - c_\gamma), \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{21} \ : \ k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \alpha\epsilon k_\mu)(c_\gamma - \alpha\epsilon k_\gamma), \quad l_{\mu\gamma} = 2(\epsilon k_\gamma c_\mu - \alpha k_\mu k_\gamma),$$
$$m_{\mu\gamma} = -k_\mu k_\gamma, \quad g_{\mu\nu\gamma} = -g_{\mu\gamma}(c_\nu - \alpha\epsilon k_\nu) - g_{\nu\gamma}(c_\mu - \alpha\epsilon k_\mu)$$
$$+2g_{\mu\nu}(c_\gamma - \alpha\epsilon k_\gamma), \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{22} \ : \ k_{\mu\gamma} = 4\omega g_{\mu\gamma} - (a_\mu - d_\mu + k_\mu\omega)(a_\gamma - d_\gamma + k_\gamma\omega),$$
$$l_{\mu\gamma} = 4\Big(2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - a_\mu a_\gamma + d_\mu d_\gamma - \omega k_\mu k_\gamma\Big),$$
$$m_{\mu\gamma} = -2k_\mu k_\gamma, \quad g_{\mu\nu\gamma} = \epsilon\Big(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu\omega) + g_{\nu\gamma}(a_\mu - d_\mu$$
$$+k_\mu\omega) - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma\omega)\Big), \quad h_{\mu\nu\gamma} = (3\epsilon/2)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma)$$
$$-\epsilon\alpha\Big(k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)\Big);$$

where $\epsilon = 1$ for $a\cdot x + d\cdot x > 0$ and $\epsilon = -1$ for $a\cdot x + d\cdot x < 0$.

5. Exact solutions of the Yang-Mills equations. When applying the symmetry reduction procedure to the nonlinear Dirac equation, we succeeded in constructing general solutions for most of the reduced systems of ODEs. In the case considered we are not so lucky. Nevertheless, we obtain some particular solutions of equations (7.1.19), (7.1.20), (7.1.22).

The principal idea of our approach to integration of systems of ODEs (7.1.19), (7.1.20), (7.1.22) is rather simple and quite natural. It is reduction of these systems by the number of components with the aid of *ad hoc* substitutions. Using this trick we have constructed particular solutions of equations 1, 2, 5, 8, 14, 15, 16, 18, 19, 20, 21, 22 ($\alpha = 0$). Below we adduce substitutions for $\vec{B}_\mu(\omega)$ and corresponding equations.

(1) $\vec{B}_\mu = a_\mu\vec{e}_1 f(\omega) + b_\mu\vec{e}_2 g(\omega) + c_\mu\vec{e}_3 h(\omega),$
$\ddot{f} - e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 - h^2)g = 0,$
$\ddot{h} + e^2(f^2 - g^2)h = 0.$

(2) $\vec{B}_\mu = b_\mu\vec{e}_1 f(\omega) + c_\mu\vec{e}_2 g(\omega) + d_\mu\vec{e}_3 h(\omega),$
$\ddot{f} + e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 + h^2)g = 0,$

$$
\begin{array}{ll}
 & \ddot{h} + e^2(f^2 + g^2)h = 0. \\
(5) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
(8.1) & (\text{under } \alpha = 0)\ \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & 4\omega \ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4\omega \ddot{g} + 4\dot{g} - \omega^{-1} g = 0. \\
(8.2) & \vec{B}_\mu = a_\mu \vec{e}_1 f(\omega) + d_\mu \vec{e}_2 g(\omega) + b_\mu \vec{e}_3 h(\omega), \\
 & 4\omega \ddot{f} + 4\dot{f} - \alpha^2 \omega^{-1} f - 2\alpha e \omega^{-1/2} g h + e^2(h^2 + g^2) f = 0, \\
 & 4\omega \ddot{g} + 4\dot{g} + \alpha^2 \omega^{-1} g + 2\alpha e \omega^{-1/2} f h + e^2(f^2 - h^2) g = 0, \\
 & 4\omega \ddot{h} + 4\dot{h} - \omega^{-1} h + 2\alpha e \omega^{-1/2} f g + e^2(f^2 - g^2) h = 0. \\
(14.1) & \vec{B}_\mu = a_\mu \vec{e}_1 f(\omega) + d_\mu \vec{e}_2 g(\omega) + c_\mu \vec{e}_3 h(\omega), \\
 & 16\ddot{f} - e^2(h^2 + g^2) f = 0, \quad 16\ddot{g} + e^2(f^2 - h^2) g = 0, \\
 & 16\ddot{h} + e^2(f^2 - g^2) h = 0. \\
(14.2) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + c_\mu \vec{e}_2 g(\omega), \\
 & 16\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
(15.1) & \vec{B}_\mu = a_\mu \vec{e}_1 f(\omega) + d_\mu \vec{e}_2 g(\omega) + (1+\alpha^2)^{-1/2}(\alpha c_\mu + b_\mu) \vec{e}_3 h(\omega), \\
 & 16(1+\alpha^2)\ddot{f} - e^2(h^2 + g^2) f = 0, \\
 & 16(1+\alpha^2)\ddot{g} + e^2(f^2 - h^2) g = 0, \\
 & 16(1+\alpha^2)\ddot{h} + e^2(f^2 - g^2) h = 0. \\
(15.2) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + (1+\alpha^2)^{-1/2}(\alpha c_\mu + b_\mu) \vec{e}_2 g(\omega), \\
 & 16(1+\alpha^2)\ddot{f} - e^2 f g^2 = 0, \quad \ddot{g} = 0. \\
(16) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & 4\omega \ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4\omega \ddot{g} + 4\dot{g} - \omega^{-1} g = 0. \\
(18) & \vec{B}_\mu = b_\mu \vec{e}_1 f(\omega) + c_\mu \vec{e}_2 g(\omega), \\
 & 4\omega \ddot{f} + 6\dot{f} + e^2 g^2 f = 0, \quad 4\omega \ddot{g} + 6\dot{g} + e^2 f^2 g = 0. \\
(19) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
(20) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
(21) & \vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \\
 & \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
(22) & (\text{under } \alpha = 0)\ \vec{B}_\mu = b_\mu \vec{e}_1 f(\omega) + c_\mu \vec{e}_2 g(\omega),
\end{array}
\tag{7.1.23}
$$

$$4\omega\ddot{f} + 8\dot{f} + e^2g^2f = 0, \quad 4\omega\ddot{g} + 8\dot{g} + e^2f^2g = 0.$$

In the above formulae we use the notations $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, $\vec{e}_3 = (0, 0, 1)$.

Thus, combining symmetry reduction by the number of independent variables and reduction by the number of dependent variables we reduce YMEs to rather simple ODEs.

Next, we will briefly consider a procedure of integration of systems of nonlinear ODEs (7.1.23).

Substitution $f = 0$, $g = h = u(\omega)$ reduces the system of ODEs 1 from (7.1.23) to the equation

$$\ddot{u} = e^2u^3, \tag{7.1.24}$$

which is integrated in elliptic functions [26, 197]. In addition, ODE (7.1.24) has a solution which is expressed in terms of elementary functions $u = \sqrt{2}(e\omega - C)^{-1}$, $C \in \mathbb{R}^1$.

ODE 2 with $f = g = h = u(\omega)$ reduces to the form $\ddot{u} + 2e^2u^3 = 0$. This equation is also integrated in elliptic functions [26, 197].

Integrating the second equation of system of ODEs 5 we get $g = C_1\omega + C_2$, $C_i \in \mathbb{R}^1$. If $C_1 \neq 0$, then the constant C_2 can be neglected, and we may put $C_2 = 0$. Provided $C_1 \neq 0$, the first equation from system 5 reads

$$\ddot{f} - e^2C_1^2\omega^2 f = 0. \tag{7.1.25}$$

The general solution of ODE (7.1.25) is given by the formula

$$f(\omega) = \omega^{1/2}Z_{1/4}\Big((ieC_1/2)\omega^2\Big).$$

Hereafter, we use the notation $Z_\nu(\omega) = C_3J_\nu(\omega) + C_4Y_\nu(\omega)$, where J_ν, Y_ν are Bessel functions, C_3, C_4 are arbitrary real constants.

In the case $C_1 = 0$, $C_2 \neq 0$ the general solution of the first equation from system 5 reads $f = C_3 \cosh C_2e\omega + C_4 \sinh C_2e\omega$, where C_3, C_4 are arbitrary real constants.

At last, provided $C_1 = C_2 = 0$, the general solution of the first equation from system 5 has the form $f = C_3\omega + C_4$, $\{C_3, C_4\} \subset \mathbb{R}^1$.

The general solution of the second ODE from system 8.1 is of the form $g = C_1\omega^{1/2} + C_2\omega^{-1/2}$, where C_1, C_2 are arbitrary real constants.

Substituting the expression obtained into the first equation we get

$$4\omega^2\ddot{f} + 4\omega\dot{f} - e^2(C_1\omega + C_2)^2 f = 0. \tag{7.1.26}$$

We cannot solve ODE (7.1.26) with $C_1C_2 \neq 0$. In the remaining cases its general solution reads

a) $C_1 \neq 0,\ C_2 = 0$

$$f = Z_0\Big((ieC_1/2)\omega\Big),$$

b) $C_1 = 0,\ C_2 \neq 0$

$$f = C_3\omega^{eC_2/2} + C_4\omega^{-eC_2/2},$$

c) $C_1 = 0,\ C_2 = 0$

$$f = C_3 \ln\omega + C_4.$$

Here $C_3,\ C_4$ are arbitrary real constants.

We did not succeed in obtaining particular solutions of the system 8.2. Equations 14.1 coincide with equations 1, if we replace e by $e/4$. Similarly, equations 14.2 coincide with equations 5, if we change e by $e/4$. Next, equations 15.1 coincide with equations 1 and equations 15.2 with equations 5, if we replace e by $(e/4)(1+\alpha^2)^{-1/2}$.

System of ODEs 16 coincides with the system 8.1 and systems 19, 20, 21 with the system 5. We did not succeed in integrating equations 18.

At last, the system 22 (under $\alpha = 0$) with the substitution $f = g = u(\omega)$ reduces to the form

$$\omega\ddot{u} + 2\dot{u} + (e^2/4)u^3 = 0. \tag{7.1.27}$$

ODE (7.1.27) is the Emden-Fowler equation which is integrated in terms of elliptic functions (see, e.g. [197]). It has two classes of particular solutions which are expressed in terms of elementary functions

$$u = e^{-1}\omega^{-1/2}, \quad u = 2\sqrt{2}C_1e^{-1}(\omega + C_1)^{-1}, \quad C_1 \in \mathbb{R}^1.$$

Substituting the results obtained into the corresponding formulae from (7.1.23) and then into the Ansatz (7.1.16), we get exact solutions of the nonlinear YMEs (7.1.1). Let us note that solutions of the systems of ODEs 5, 8.1, 14.2, 15.2, 16, 19, 20, 21 satisfying the condition $g = 0$ give rise to Abelian solutions of YMEs. We do not adduce these and present non-Abelian solutions of YMEs only.

1) $\vec{A}_\mu = (\vec{e}_2b_\mu + \vec{e}_3c_\mu)\sqrt{2}(ed\cdot x - \lambda)^{-1};$

2) $\vec{A}_\mu = (\vec{e}_2b_\mu + \vec{e}_3c_\mu)\lambda\,\mathrm{sn}\,[(e\sqrt{2}/2)\lambda d\cdot x]\,\mathrm{dn}\,[(e\sqrt{2}/2)\lambda d\cdot x]$
$\times\Big(\mathrm{cn}\,[(e\sqrt{2}/2)\lambda d\cdot x]\Big)^{-1};$

3) $\vec{A}_\mu = (\vec{e}_2b_\mu + \vec{e}_3c_\mu)\lambda\Big(\mathrm{cn}\,(e\lambda d\cdot x)\Big)^{-1};$

4) $\vec{A}_\mu = (\vec{e}_1 b_\mu + \vec{e}_2 c_\mu + \vec{e}_3 d_\mu)\lambda\,\mathrm{cn}\,(e\sqrt{2}\lambda a\cdot x);$

5) $\vec{A}_\mu = \vec{e}_1 k_\mu |k\cdot x|^{-1}(c\cdot x)^{1/2} Z_{1/4}\Big((ie\lambda/2)(c\cdot x)^2\Big) + \vec{e}_2 b_\mu \lambda c\cdot x;$

6) $\vec{A}_\mu = \vec{e}_1 k_\mu |k\cdot x|^{-1}\Big(\lambda_1 \cosh(e\lambda c\cdot x) + \lambda_2 \sinh(e\lambda c\cdot x)\Big) + \vec{e}_2 b_\mu \lambda;$

7) $\vec{A}_\mu = \vec{e}_1 k_\mu Z_0\Big((ie\lambda/2)[(b\cdot x)^2 + (c\cdot x)^2]\Big) + \vec{e}_2(b_\mu c\cdot x - c_\mu b\cdot x)\lambda;$

8) $\vec{A}_\mu = \vec{e}_1 k_\mu\Big(\lambda_1[(b\cdot x)^2 + (c\cdot x)^2]^{e\lambda/2} + \lambda_2[(b\cdot x)^2 + (c\cdot x)^2]^{-e\lambda/2}\Big)$
$+\vec{e}_2(b_\mu c\cdot x - c_\mu b\cdot x)\lambda[(b\cdot x)^2 + (c\cdot x)^2]^{-1};$

9) $\vec{A}_\mu = \Big\{\vec{e}_2\Big((1/8)[d_\mu - k_\mu(k\cdot x)^2] + (1/2)b_\mu k\cdot x\Big) + \vec{e}_3 c_\mu\Big\}\lambda$
$\times\mathrm{sn}\,\Big((e\lambda\sqrt{2}/8)[4b\cdot x + (k\cdot x)^2]\Big)\,\mathrm{dn}\,\Big((e\lambda\sqrt{2}/8)[4b\cdot x + (k\cdot x)^2]\Big)$
$\times\Big\{\mathrm{cn}\,\Big((e\lambda\sqrt{2}/8)[4b\cdot x + (k\cdot x)^2]\Big)\Big\}^{-1};$

10) $\vec{A}_\mu = \Big\{\vec{e}_2\Big((1/8)[d_\mu - k_\mu(k\cdot x)^2] + (1/2)b_\mu k\cdot x\Big) + \vec{e}_3 c_\mu\Big\}\lambda$
$\times\Big\{\mathrm{cn}\,\Big((e\lambda/4)[4b\cdot x + (k\cdot x)^2]\Big)\Big\}^{-1};$

11) $\vec{A}_\mu = \Big\{\vec{e}_2\Big((1/8)[d_\mu - k_\mu(k\cdot x)^2] + (1/2)b_\mu k\cdot x\Big) + \vec{e}_3 c_\mu\Big\}4\sqrt{2}$
$\times\Big(e[4b\cdot x + (k\cdot x)^2] - \lambda\Big)^{-1};$

12) $\vec{A}_\mu = \vec{e}_1 k_\mu[4b\cdot x + (k\cdot x)^2]^{1/2} Z_{1/4}\Big((ie\lambda/8)[4b\cdot x + (k\cdot x)^2]^2\Big)$
$+\vec{e}_2 c_\mu \lambda[4b\cdot x + (k\cdot x)^2];$

13) $\vec{A}_\mu = \vec{e}_1 k_\mu\Big\{\lambda_1\cosh\Big((e\lambda/4)[4b\cdot x + (k\cdot x)^2]\Big) + \lambda_2\sinh\Big((e\lambda/4)[4b\cdot x$
$+(k\cdot x)^2]\Big)\Big\} + \vec{e}_2 c_\mu \lambda;$

14) $\vec{A}_\mu = \Big(\vec{e}_2[d_\mu - (1/8)k_\mu(k\cdot x)^2 - (1/2)b_\mu k\cdot x] + \vec{e}_3[\alpha c_\mu + b_\mu$
$+(1/2)k_\mu k\cdot x](1+\alpha^2)^{-1/2}\Big\}\lambda\,\mathrm{sn}\,\Big((e\lambda\sqrt{2}/8)[4(\alpha b\cdot x - c\cdot x)$
$+\alpha(k\cdot x)^2](1+\alpha^2)^{-1/2}\Big)\,\mathrm{dn}\,\Big((e\lambda\sqrt{2}/8)[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]$
$\times(1+\alpha^2)^{-1/2}\Big)\Big\{\mathrm{cn}\,\Big((e\lambda\sqrt{2}/8)[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]$
$\times(1+\alpha^2)^{-1/2}\Big)\Big\}^{-1};$

15) $\vec{A}_\mu = \Big(\vec{e}_2[d_\mu - (1/8)k_\mu(k\cdot x)^2 - (1/2)b_\mu k\cdot x] + \vec{e}_3[\alpha c_\mu + b_\mu$
$+(1/2)k_\mu k\cdot x](1+\alpha^2)^{-1/2}\Big)\Big\{\mathrm{cn}\,\Big((e\lambda/4)[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]$

$$\times(1+\alpha^2)^{-1/2}\Big)\Big\}^{-1};$$

16) $$\vec{A}_\mu = \Big(\vec{e}_2[d_\mu - (1/8)k_\mu(k\cdot x)^2 - (1/2)b_\mu k\cdot x] + \vec{e}_3[\alpha c_\mu + b_\mu + (1/2)k_\mu k\cdot x](1+\alpha^2)^{-1/2}\Big)4\sqrt{2}(1+\alpha^2)^{1/2}\Big(e[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]\Big)^{-1}; \tag{7.1.28}$$

17) $$\vec{A}_\mu = \vec{e}_1 k_\mu\Big\{[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]^{1/2} Z_{1/4}\Big((ie\lambda/8)[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]^2(1+\alpha^2)^{-1/2}\Big)\Big\} + \vec{e}_2[\alpha c_\mu + b_\mu + (1/2)k_\mu k\cdot x]\lambda \times[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2](1+\alpha^2)^{-1/2};$$

18) $$\vec{A}_\mu = \vec{e}_1 k_\mu\Big\{\lambda_1\cosh\Big((e\lambda/4)(1+\alpha^2)^{-1/2}[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]\Big) + \lambda_2\sinh\Big((e\lambda/4)(1+\alpha^2)^{-1/2}[4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2]\Big)\Big\} + \vec{e}_2[\alpha c_\mu + b_\mu + (1/2)k_\mu k\cdot x]\lambda(1+\alpha^2)^{-1/2};$$

19) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1} Z_0\Big((ie\lambda/2)[(b\cdot x)^2 + (c\cdot x)^2]\Big) + \vec{e}_2(b_\mu c\cdot x - c_\mu b\cdot x)\lambda;$$

20) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}\Big(\lambda_1[(b\cdot x)^2 + (c\cdot x)^2]^{e\lambda/2} + \lambda_2[(b\cdot x)^2 + (c\cdot x)^2]^{-e\lambda/2}\Big) + \vec{e}_2(b_\mu c\cdot x - c_\mu b\cdot x)\lambda[(b\cdot x)^2 + (c\cdot x)^2]^{-1};$$

21) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}(c\cdot x)^{1/2} Z_{1/4}[(ie\lambda/2)(c\cdot x)^2] + \vec{e}_2[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}]\lambda c\cdot x;$$

22) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}\Big(\lambda_1\cosh(\lambda e c\cdot x) + \lambda_2\sinh(\lambda e c\cdot x)\Big) + \vec{e}_2[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}]\lambda;$$

23) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}(\ln|k\cdot x| - c\cdot x)^{1/2} Z_{1/4}[(ie\lambda/2)(\ln|k\cdot x| - c\cdot x)^2] + \vec{e}_2[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}]\lambda(\ln|k\cdot x| - c\cdot x);$$

24) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}\Big(\lambda_1\cosh[\lambda e(\ln|k\cdot x| - c\cdot x)] + \lambda_2\sinh[\lambda e(\ln|k\cdot x| - c\cdot x)]\Big) + \vec{e}_2[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}]\lambda;$$

25) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}(\alpha\ln|k\cdot x| - c\cdot x)^{1/2} Z_{1/4}[(ie\lambda/2)(\alpha\ln|k\cdot x| - c\cdot x)^2] + \vec{e}_2[b_\mu - k_\mu(b\cdot x - \ln|k\cdot x|)(k\cdot x)^{-1}]\lambda(\alpha\ln|k\cdot x| - c\cdot x);$$

26) $$\vec{A}_\mu = \vec{e}_1 k_\mu|k\cdot x|^{-1}\Big(\lambda_1\cosh[\lambda e(\alpha\ln|k\cdot x| - c\cdot x)] + \lambda_2\sinh[\lambda e \times(\alpha\ln|k\cdot x| - c\cdot x))\Big) + \vec{e}_2[b_\mu - k_\mu(b\cdot x - \ln|k\cdot x|)(k\cdot x)^{-1}]\lambda;$$

$$27)\quad \vec{A}_\mu = \Big(\vec{e}_1[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}] + \vec{e}_2[c_\mu - k_\mu c\cdot x(k\cdot x)^{-1}]\Big)e^{-1} \times \begin{cases} (x\cdot x)^{-1/2}, \\ 2\sqrt{2}\lambda(x\cdot x+\lambda)^{-1}; \end{cases}$$

$$28)\quad \vec{A}_\mu = \Big(\vec{e}_1[b_\mu - k_\mu b\cdot x(k\cdot x)^{-1}] + \vec{e}_2[c_\mu - k_\mu c\cdot x(k\cdot x)^{-1}]\Big)f(x\cdot x),$$
$$\omega\ddot{f} + 2\dot{f} + (e^2 f^3/4) = 0.$$

In the above formulae $Z_\alpha(\omega)$ is the Bessel function; sn, dn, cn are Jacobi elliptic functions having the modulus $\sqrt{2}/2$; $\lambda,\ \lambda_1,\ \lambda_2 = \mathrm{const}$.

Let us note that the solutions N 27 are nothing more but the meron and the instanton solutions of YMEs [2]. In the Euclidean space the meron and instanton solutions were obtained by Alfaro, Fubini, Furlan [68] and Belavin, Polyakov, Schwartz, Tyupkin [29] with the use of the Ansatz suggested by 't Hooft [278], Corrigan and Fairlie [60] and Wilczek [285].

Another important point is that we can obtain new exact solutions of YMEs by applying to solutions (7.1.28) the solution generation technique. We do not adduce the corresponding formulae because of their awkwardness.

6. Some generalizations. It was noticed in [157, 158] that group-invariant solutions of nonlinear PDEs could provide us with rather general information about the structure of solutions of the equation under study. Using this fact, we constructed in [157, 158, 160] a number of new exact solutions of the nonlinear Dirac equation which could not be obtained by the symmetry reduction procedure (see also Sections 6.1 and 7.2). We will demonstrate that the same idea proves to be efficient for constructing new solutions of YMEs.

Solutions of YMEs numbered by 7, 8, 19, 20 can be represented in the following unified form:

$$\vec{A}_\mu = k_\mu \vec{B}(k\cdot x,\ c\cdot x) + b_\mu \vec{C}(k\cdot x,\ c\cdot x). \tag{7.1.29}$$

Substituting the Ansatz (7.1.29) into YMEs and splitting the equality obtained with respect to linearly independent four-vectors with components k_μ, b_μ, c_μ, we get

$$\begin{aligned} &1)\quad \vec{C}_{\omega_1\omega_1} = \vec{0}, \\ &2)\quad \vec{C}\times\vec{C}_{\omega_1} = \vec{0}, \\ &3)\quad \vec{B}_{\omega_1\omega_1} + e\vec{C}_{\omega_0}\times\vec{C} + e^2\vec{C}\times(\vec{C}\times\vec{B}) = \vec{0}. \end{aligned} \tag{7.1.30}$$

Here we use the notations $\omega_0 = k\cdot x,\ \omega_1 = c\cdot x$.

The general solution of the first two equations from (7.1.30) is given by one of the formulae

I. $\vec{C} = \vec{f}(\omega_0)$,

II. $\vec{C} = \Big(\omega_1 + v_0(\omega_0)\Big)\vec{f}(\omega_0)$,

where $v_0,\ \vec{f}$ are arbitrary smooth functions.

Consider the case $\vec{C} = \vec{f}(\omega_0)$. Substituting this expression into the third equation from (7.1.30) we have

$$\vec{B}_{\omega_1\omega_1} + e\vec{f}_{\omega_0} \times \vec{f} + e^2\vec{f}(\vec{f}\vec{B}) - e^2\vec{f}^{\,2}\vec{B} = \vec{0}. \tag{7.1.31}$$

Since equations (7.1.31) do not contain derivatives of $\vec{B}$ with respect to ω_0, they can be considered as a system of ODEs with respect to the variable ω_1. Multiplying (7.1.31) by $\vec{f}$ we arrive at the relation $(\vec{B}\vec{f}\,)_{\omega_1\omega_1} = 0$, whence

$$\vec{B}\vec{f} = v_1(\omega_0)\omega_1 + v_2(\omega_0). \tag{7.1.32}$$

In (7.1.32) $v_1,\ v_2$ are arbitrary sufficiently smooth functions.

With account of (7.1.32) system (7.1.31) reads

$$\vec{B}_{\omega_1\omega_1} - e^2\vec{f}^{\,2}\vec{B} = e\vec{f} \times \vec{f}_{\omega_0} - e^2(v_1\omega_1 + v_2)\vec{f}.$$

The above linear system of ODEs is easily integrated. Its general solution is given by the formula

$$\begin{aligned}\vec{B} = {}& \vec{g}(\omega_0)\cosh e|\vec{f}|\omega_1 + \vec{h}(\omega_0)\sinh e|\vec{f}|\omega_1 + e^{-1}|\vec{f}|^{-2}\vec{f}_{\omega_0} \times \vec{f} \\ & \times|\vec{f}|^{-2}(v_1\omega_1 + v_2)\vec{f},\end{aligned} \tag{7.1.33}$$

where $\vec{g},\ \vec{h}$ are arbitrary smooth functions.

Substituting (7.1.33) into (7.1.32) we get the following restrictions on the choice of the functions $\vec{g},\ \vec{h}$:

$$\vec{f}\vec{g} = 0, \quad \vec{f}\vec{h} = 0. \tag{7.1.34}$$

Thus, provided $\vec{C}_{\omega_1} = 0$, the general solution of the system of ODEs (7.1.31) is given by formulae (7.1.33), (7.1.34). Substituting (7.1.33) into the initial Ansatz (7.1.29) we obtain the following family of exact solutions of YMEs:

$$\begin{aligned}\vec{A}_\mu \;\; = \;\; & k_\mu\Big\{\vec{g}\cosh e|\vec{f}|c\cdot x + \vec{h}\sinh e|\vec{f}|c\cdot x + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} \\ & +(v_1 c\cdot x + v_2)\vec{f}\Big\} + b_\mu\vec{f},\end{aligned}$$

where $\vec{f},\ \vec{g},\ \vec{h},\ v_1,\ v_2$ are arbitrary smooth functions of $k\cdot x$ satisfying (7.1.34), an overdot denotes differentiation with respect to $\omega_0 = k\cdot x$.

The case $\vec{C} = [\omega_1 + v_0(\omega_0)]\vec{f}(\omega_0)$ is treated in a similar way. As a result, we obtain the following family of exact solutions of YMEs:

$$\begin{aligned}\vec{A}_\mu &= k_\mu\Big\{(c\cdot x+v_0)^{1/2}\Big(\vec{g}J_{1/4}[(ie/2)|\vec{f}|(c\cdot x+v_0)^2]\\ &\quad+\vec{h}Y_{1/4}[(ie/2)|\vec{f}|(c\cdot x+v_0)^2]\Big)+(v_1c\cdot x+v_2)\vec{f}\\ &\quad+e^{-1}|\vec{f}|^{-2}\dot{\vec{f}}\times\vec{f}\Big\}+b_\mu(c\cdot x+v_0)\vec{f},\end{aligned}$$

where $\vec{f},\ \vec{g},\ \vec{h},\ v_0,\ v_1,\ v_2$ are arbitrary smooth functions of $k\cdot x$ satisfying (7.1.34), $J_{1/4}(\omega),\ Y_{1/4}(\omega)$ are the Bessel functions.

Another effective Ansatz for the Yang-Mills field is obtained if we replace $c\cdot x$ in (7.1.29) by $b\cdot x$

$$\vec{A}_\mu = k_\mu\vec{B}(k\cdot x,\, b\cdot x)+b_\mu\vec{C}(k\cdot x,\, b\cdot x). \tag{7.1.35}$$

Substitution of (7.1.35) into YMEs yields the following system of PDEs for $\vec{B},\ \vec{C}$:

$$\vec{B}_{\omega_1\omega_1}-\vec{C}_{\omega_0\omega_1}-e(\vec{B}\times\vec{C}_{\omega_1}+2\vec{B}_{\omega_1}\times\vec{C}+\vec{C}\times\vec{C}_{\omega_0})+e^2\vec{C}\times(\vec{C}\times\vec{B})=\vec{0}. \tag{7.1.36}$$

We have succeeded in integrating system (7.1.36), provided $\vec{C} = \vec{f}(\omega_0)$. Substituting the result obtained into (7.1.35), we come to the following family of exact solutions of YMEs:

$$\begin{aligned}\vec{A}_\mu &= k_\mu\Big\{(\vec{g}+b\cdot x|\vec{f}|^{-1}\vec{g}\times\vec{f})\cos(e|\vec{f}|b\cdot x)+(\vec{h}+b\cdot x|\vec{f}|^{-1}\vec{h}\times\vec{f})\\ &\quad\times\sin(e|\vec{f}|b\cdot x)+e^{-1}|\vec{f}|^{-2}\dot{\vec{f}}\times\vec{f}+(v_1b\cdot x+v_2)\vec{f}\Big\}+b_\mu\vec{f},\end{aligned}$$

where $\vec{f},\ \vec{g},\ \vec{h},\ v_1,\ v_2$ are arbitrary smooth functions of $k\cdot x$.

In addition, we have constructed the following class of exact solutions of YMEs:

$$\vec{A}_\mu = k_\mu\vec{e}_1vu^2(b\cdot x)+b_\mu\vec{e}_2u(b\cdot x),$$

where $\vec{e}_1=(1,0,0),\ \vec{e}_2=(0,1,0);\ v$ is an arbitrary smooth function of $k\cdot x$; $u(b\cdot x)$ is a solution of the nonlinear ODE $\ddot{u}=e^2u^5$, which is integrated in elliptic functions.

In conclusion of this section we will obtain a generalization of the plane-wave Coleman solution [54]

$$\vec{A}_\mu = k_\mu\Big(\vec{f}(k\cdot x)b\cdot x + \vec{g}(k\cdot x)c\cdot x\Big). \tag{7.1.37}$$

It is not difficult to verify that (7.1.37) satisfy YMEs with arbitrary $\vec{f}$, $\vec{g}$. Evidently, solution (7.1.37) is a particular case of the Ansatz

$$\vec{A}_\mu = k_\mu \vec{B}(k\cdot x,\, b\cdot x,\, c\cdot x). \tag{7.1.38}$$

Substituting (7.1.38) into YMEs we get

$$\vec{B}_{\omega_1\omega_1} + \vec{B}_{\omega_2\omega_2} = \vec{0}, \tag{7.1.39}$$

where $\omega_1 = b\cdot x,\ \omega_2 = c\cdot x$.

Integrating the Laplace equations (7.1.39) and substituting the result obtained into (7.1.38) we have

$$\vec{A}_\mu = k_\mu\Big(\vec{U}(k\cdot x,\, b\cdot x + ic\cdot x) + \vec{U}(k\cdot x,\, b\cdot x - ic\cdot x)\Big).$$

Here $\vec{U}(k\cdot x,\, z)$ is an arbitrary analytical with respect to z function. Choosing $\vec{U} = (1/2)[\vec{f}(k\cdot x) - i\vec{g}(k\cdot x)]z$ we get the Coleman's solution (7.1.37).

7.2. Non-Lie reduction of the Yang-Mills equations

In the present section we will obtain conditionally-invariant Ansätze for the Yang-Mills field $\vec{A}_\mu(x)$ utilizing the idea which enables us to construct non-Lie (conditionally-invariant) Ansätze for the spinor field $\psi(x)$. This idea proves to be fruitful for obtaining new reductions and constructing new exact solutions of the $SU(2)$ Yang-Mills equations (7.1.1) as compared with those found by means of the symmetry reduction of YMEs.

1. Reduction of YMEs. We are looking for a solution of YMEs of the form (7.1.11), (7.1.16) without imposing *a priori* conditions on the functions $\omega(x)$, $\theta_\mu(x)$. They should be determined from the requirement that substitution of the Ansatz (7.1.11) into system of PDEs (7.1.1) yields a system of ordinary differential equations for a vector function $\vec{B}_\mu(\omega)$.

By direct check one can become convinced of that the following assertion holds true.

Lemma 7.2.1. *Ansatz (7.1.11), (7.1.16) reduces YMEs (7.1.1) to a system of ODEs iff the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the system of PDEs*

$$\begin{array}{ll} 1) & \omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \\ 2) & \Box\omega = F_2(\omega), \\ 3) & Q_{\alpha\mu}\omega_{x_\alpha} = G_\mu(\omega), \\ 4) & Q_{\alpha\mu x_\alpha} = H_\mu(\omega), \\ 5) & Q^\alpha_\mu Q_{\alpha\nu x_\beta}\omega_{x^\beta} = R_{\mu\nu}(\omega), \\ 6) & Q^\alpha_\mu \Box Q_{\alpha\nu} = S_{\mu\nu}(\omega), \\ 7) & Q^\alpha_\mu Q_{\alpha\nu x_\beta}Q_{\beta\gamma} + Q^\alpha_\nu Q_{\alpha\gamma x_\beta}Q_{\beta\mu} + Q^\alpha_\gamma Q_{\alpha\mu x_\beta}Q_{\beta\nu} = T_{\mu\nu\gamma}(\omega), \end{array} \tag{7.2.1}$$

where $F_1,\ F_2,\ G_\mu,\ldots,T_{\mu\nu\gamma}$ are some smooth functions, $\mu,\nu,\gamma = 0,\ldots,3$.

And what is more, a reduced equation has the form

$$\begin{aligned} & k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eq_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma \\ & \quad + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}, \end{aligned} \tag{7.2.2}$$

where

$$\begin{array}{rcl} k_{\mu\gamma} & = & g_{\mu\gamma}F_1 - G_\mu G_\gamma, \\ l_{\mu\gamma} & = & g_{\mu\gamma}F_2 + 2R_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ m_{\mu\gamma} & = & S_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\ q_{\mu\nu\gamma} & = & g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ h_{\mu\nu\gamma} & = & (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}. \end{array} \tag{7.2.3}$$

Thus, to describe all Ansätze of the form (7.1.11) reducing YMEs to a system of ODEs we have to construct the general solution of the over-determined system of PDEs (7.1.16), (7.2.1). Let us emphasize that system (7.1.16), (7.2.1) is compatible since Ansätze for the Yang-Mills field $\vec{Y}_\mu(x)$ invariant under the $P(1,3)$ non-conjugate subgroups of the Poincaré group satisfy equations (7.1.16), (7.2.1) with some specific choice of the functions $F_1,\ F_2,\ldots,T_{\mu\nu\gamma}$.

Computations needed to integrate system of nonlinear PDEs (7.1.16), (7.2.1) are rather involved. In addition, they have much in common with those performed to obtain conditionally-invariant Ansätze for the spinor field

(Theorem 6.1.1). That is why we present here only a principal idea of our approach to solving the system (7.1.16), (7.2.1). When integrating it we use essentially the fact that the general solution of system of equations 1, 2 from (7.2.1) is known (see Section 2.1). With already known $\omega(x)$ we proceed to integration of linear PDEs 3, 4 from (7.2.1). Next, we substitute the results obtained into the remaining equations and thus get the final form of the functions $\omega(x)$, $\theta_\mu(x)$.

Before adducing the results of integration of system of PDEs (7.1.16), (7.2.1) we make a remark. As a direct check shows, the structure of the Ansatz (7.1.11), (7.1.16) is not altered by the change of variables

$$\begin{aligned}
&\omega \to \omega' = T(\omega), \quad \theta_0 \to \theta_0' = \theta_0 + T_0(\omega),\\
&\theta_1 \to \theta_1' = \theta_1 + e^{\theta_0}\Big(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3\Big),\\
&\theta_2 \to \theta_2' = \theta_2 + e^{\theta_0}\Big(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3\Big),\\
&\theta_3 \to \theta_3' = \theta_3 + T_3(\omega),
\end{aligned} \tag{7.2.4}$$

where $T(\omega)$, $T_\mu(\omega)$ are arbitrary smooth functions. That is why solutions of system (7.1.16), (7.2.1) connected by the relations (7.2.4) are considered as equivalent.

It occurs that the new (non-Lie) Ansätze are obtained only when the functions $\omega(x)$, $\theta_\mu(x)$ up to the equivalence relations (7.2.4) have the form

$$\begin{aligned}
&\theta_\mu = \theta_\mu(\xi,\, b\cdot x,\, c\cdot x),\\
&\omega = \omega(\xi,\, b\cdot x,\, c\cdot x),
\end{aligned} \tag{7.2.5}$$

where $\xi = (1/2)k\cdot x$, $k_\nu = a_\nu + d_\nu$, $\mu,\nu = 0,\ldots,3$.

A list of inequivalent solutions of system of PDEs (7.1.16), (7.2.1) belonging to the class (7.2.5) is exhausted by the following solutions:

$$\begin{aligned}
1)\quad &\theta_0 = \theta_3 = 0, \quad \omega = (1/2)k\cdot x, \quad \theta_1 = w_0(\xi)b\cdot x + w_1(\xi)c\cdot x,\\
&\theta_2 = w_2(\xi)b\cdot x + w_3(\xi)c\cdot x;\\[1ex]
2)\quad &\omega = b\cdot x + w_1(\xi), \quad \theta_0 = \alpha\Big(c\cdot x + w_2(\xi)\Big),\\
&\theta_a = -(1/4)\dot w_a(\xi), \quad a = 1,2, \quad \theta_3 = 0,\\[1ex]
3)\quad &\theta_0 = T(\xi), \quad \theta_3 = w_1(\xi), \quad \omega = b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2(\xi),\\
&\theta_1 = \Big((1/4)(\varepsilon e^T + \dot T)(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\Big)\sin w_1
\end{aligned} \tag{7.2.6}$$

$$+(1/4)\Big(\dot{w}_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot{w}_2\Big)\cos w_1,$$

$$\theta_2 = -\Big((1/4)(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\Big)\cos w_1$$

$$+(1/4)\Big(\dot{w}_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot{w}_2\Big)\sin w_1;$$

4) $\theta_0 = 0, \quad \theta_3 = \arctan\Big([c\cdot x + w_2(\xi)][b\cdot x + w_1(\xi)]^{-1}\Big),$

$\theta_a = -(1/4)\dot{w}_a(\xi), \quad a = 1, 2,$

$\omega = \Big([b\cdot x + w_1(\xi)]^2 + [c\cdot x + w_2(\xi)]^2\Big)^{1/2}.$

Here $\alpha \neq 0$ is an arbitrary constant, $\varepsilon = \pm 1$, $w_0,\ w_1,\ w_2,\ w_3$ are arbitrary smooth functions of $\xi = (1/2)k\cdot x$, $T = T(\xi)$ is a solution of the nonlinear ODE

$$(\dot{T} + \varepsilon e^T)^2 + \dot{w}_1^2 = \varkappa e^{2T}, \quad \varkappa \in \mathbb{R}^1. \tag{7.2.7}$$

Substitution of the Ansatz (7.1.11), where $Q_{\mu\nu}(x)$ are given by formulae (7.1.16), (7.2.6), into YMEs yields systems of nonlinear ODEs of the form (7.2.2), where

1) $k_{\mu\gamma} = -(1/4)k_\mu k_\gamma, \quad l_{\mu\gamma} = -(w_0 + w_3)k_\mu k_\gamma,$

$m_{\mu\gamma} = -4\ (w_0^2 + w_1^2 + w_2^2 + w_3^2)k_\mu k_\gamma - (\dot{w}_0 + \dot{w}_3)k_\mu k_\gamma,$

$q_{\mu\nu\gamma} = (1/2)(g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma),$

$h_{\mu\nu\gamma} = (w_0 + w_3)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + 2(w_1 - w_2)\Big((k_\mu b_\nu - k_\nu b_\mu)\ c_\gamma$

$+(b_\mu c_\nu - b_\nu c_\mu)k_\gamma + (c_\mu k_\nu - c_\nu k_\mu)b_\gamma\Big);$

2) $k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma),$

$q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,$

$h_{\mu\nu\gamma} = \alpha\Big((a_\mu d_\nu - a_\nu d_\mu)c_\gamma + (d_\mu c_\nu - d_\nu c_\mu)a_\gamma + (c_\mu a_\nu - c_\nu a_\mu)d_\gamma\Big);$

3) $k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -(\varepsilon/2)b_\mu k_\gamma,$ (7.2.8)

$m_{\mu\gamma} = -(\varkappa/4)k_\mu k_\gamma, \quad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,$

$h_{\mu\nu\gamma} = (\varepsilon/4)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$

4) $k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\omega^{-1}(g_{\mu\gamma} + b_\mu b_\gamma),$

$m_{\mu\gamma} = -\omega^{-2}c_\mu c_\gamma, \quad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,$

$h_{\mu\nu\gamma} = (1/2)\omega^{-1}(g_{\mu\gamma}b_\nu - g_{\mu\nu}b_\gamma).$

2. Exact solutions of the Yang-Mills equations. Systems (7.2.2), (7.2.8) are systems of twelve nonlinear second-order ODEs with variable coefficients. That is why there is a little hope to construct their general solutions. But it is possible to obtain particular solutions of system (7.2.2) with coefficients given by formulae 2–4 from (7.2.8).

Consider, as an example, system of ODEs (7.2.2) with coefficients given by the formulae 2 from (7.2.8). We look for its solutions in the form

$$\vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \quad fg \neq 0, \tag{7.2.9}$$

where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$.

Substituting the expression (7.2.9) into the above mentioned system we get

$$\ddot{f} + (\alpha^2 - e^2 g^2) f = 0, \quad f\dot{g} + 2\dot{f} g = 0. \tag{7.2.10}$$

The second ODE from (7.2.10) is easily integrated

$$g = \lambda f^{-2}, \quad \lambda \in \mathbb{R}^1, \quad \lambda \neq 0. \tag{7.2.11}$$

Substitution of the result obtained into the first ODE from (7.2.10) yields the Ermakov-type equation for $f(\omega)$

$$\ddot{f} + \alpha^2 f - e^2 \lambda^2 f^{-3} = 0,$$

which is integrated in elementary functions [197]

$$f = \Big(\alpha^{-2} C^2 + \alpha^{-2} (C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \sin 2|\alpha|\omega\Big)^{1/2}. \tag{7.2.12}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (7.2.9), (7.2.11), (7.2.12) into the corresponding Ansatz for $\vec{A}_\mu(x)$ we get the following class of exact solutions of YMEs (7.1.1):

$$\begin{aligned}
\vec{A}_\mu \quad = \quad & \vec{e}_1 k_\mu \exp\,(-\alpha c\cdot x - \alpha w_2)\Big(\alpha^{-2} C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \\
& \times \sin 2|\alpha|(b\cdot x + w_1)\Big)^{1/2} + \vec{e}_2 \lambda \Big(\alpha^{-2} C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \\
& \times \sin 2|\alpha|(b\cdot x + w_1)\Big)^{-1} \Big(b_\mu + (1/2) k_\mu \dot{w}_1\Big).
\end{aligned}$$

In a similar way we have obtained five other classes of the exact solutions of the Yang-Mills equations

$$\vec{A}_\mu \quad = \quad \vec{e}_1 k_\mu e^{-T} (b\cdot x \cos w_1 + c\cdot x \sin w_1 + w_2)^{1/2} Z_{1/4}\Big((ie\lambda/2)(b\cdot x \cos w_1$$

$$
\begin{aligned}
&+c\cdot x\sin w_1+w_2)^2\Big)+\vec{e}_2\lambda\ (b\cdot x\cos w_1+c\cdot x\sin w_1+w_2)\\
&\times\Big(c_\mu\cos w_1-b_\mu\sin w_1+2k_\mu[(1/4)(\varepsilon e^T+\dot{T})(b\cdot x\sin w_1\\
&-c\cdot x\cos w_1)+w_3]\Big);
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}_\mu &= \vec{e}_1k_\mu e^{-T}\Big(C_1\cosh[e\lambda(b\cdot x\cos w_1+c\cdot x\sin w_1+w_2)]+C_2\sinh[e\lambda\\
&\times(b\cdot x\cos w_1+c\cdot x\sin w_1+w_2)]\Big)+\vec{e}_2\lambda\Big(c_\mu\cos w_1-b_\mu\sin w_1\\
&+2k_\mu[(1/4)(\varepsilon e^T+\dot{T})(b\cdot x\sin w_1-c\cdot x\cos w_1)+w_3]\Big);
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}_\mu &= \vec{e}_1k_\mu e^{-T}\Big(C^2(b\cdot x\cos w_1+c\cdot x\sin w_1+w_2)^2+\lambda^2e^2C^{-2}\Big)^{1/2}\\
&+\vec{e}_2\lambda\Big(C^2(b\cdot x\cos w_1+c\cdot x\sin w_1+w_2)^2+\lambda^2e^2C^{-2}\Big)^{-1}\\
&\times\Big(b_\mu\cos w_1+c_\mu\sin w_1-(1/2)k_\mu[\dot{w}_1(b\cdot x\sin w_1\\
&-c\cdot x\cos w_1)-\dot{w}_2]\Big);
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}_\mu &= \vec{e}_1k_\mu Z_0\Big((ie\lambda/2)[(b\cdot x+w_1)^2+(c\cdot x+w_2)^2]\Big)+\vec{e}_2\lambda\Big(c_\mu(b\cdot x+w_1)\\
&-b_\mu(c\cdot x+w_2)-(1/2)k_\mu[\dot{w}_1(c\cdot x+w_2)-\dot{w}_2(b\cdot x+w_1)]\Big);
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}_\mu &= \vec{e}_1k_\mu\Big(C_1[(b\cdot x+w_1)^2+(c\cdot x+w_2)^2]^{e\lambda/2}+C_2[(b\cdot x+w_1)^2\\
&+(c\cdot x+w_2)^2]^{-e\lambda/2}\Big)+\vec{e}_2\lambda[(b\cdot x+w_1)^2+(c\cdot x+w_2)^2]^{-1}\\
&\times\Big(c_\mu(b\cdot x+w_1)-b_\mu(c\cdot x+w_2)-(1/2)k_\mu[\dot{w}_1(c\cdot x+w_2)\\
&-\dot{w}_2(b\cdot x+w_1)]\Big).
\end{aligned}
$$

Here $C_1,\ C_2,\ C\neq 0,\ \lambda$ are arbitrary parameters; $w_1,\ w_2,\ w_3$ are arbitrary smooth functions of $\xi=(1/2)k\cdot x$, $T=T(\xi)$ is a solution of ODE (7.2.7) and

$$
\begin{aligned}
&Z_s(\omega)=C_1J_s(\omega)+C_2Y_s(\omega),\\
&\vec{e}_1=(1,0,0),\quad \vec{e}_2=(0,1,0),
\end{aligned}
$$

where $J_s,\ Y_s$ are Bessel functions.

Thus, we have obtained broad families of exact non-Abelian solutions of YMEs (7.1.1).

In conclusion of the section we will say a few words about the symmetry interpretation of the Ansätze (7.1.11), (7.1.16), (7.2.6). Let us consider as an

example the Ansatz determined by the formulae 1 from (7.2.6). As a direct computation shows, the generators of a three-parameter Lie group leaving it invariant are of the form

$$
\begin{aligned}
Q_1 &= k_\alpha \partial_\alpha, \\
Q_2 &= b_\alpha \partial_\alpha - 2\left\{ [w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)] \sum_{a=1}^{3} A^{a\nu} \right\} \partial_{A^{a\mu}}, \\
Q_3 &= c_\alpha \partial_\alpha - 2\left\{ [w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)] \sum_{a=1}^{3} A^{a\nu} \right\} \partial_{A^{a\mu}}.
\end{aligned}
$$

Evidently, system of PDEs (7.1.1) is invariant under the one-parameter group having the generator Q_1. But it is not invariant under the groups having the generators Q_2, Q_3. At the same time, the system of PDEs

$$
\begin{aligned}
& \partial_\nu \partial^\nu \vec{A}_\mu - \partial^\mu \partial_\nu \vec{A}_\nu + e\Big((\partial_\nu \vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_\nu \vec{A}_\mu) \times \vec{A}_\nu \\
& \quad + (\partial^\mu \vec{A}_\nu) \times \vec{A}^\nu \Big) + e^2 \vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}, \\
& Q_0 \vec{A}_\mu \equiv k_\alpha \partial_\alpha \vec{A}_\mu = \vec{0}, \\
& Q_1 \vec{A}_\mu \equiv b_\alpha \partial_\alpha \vec{A}_\mu + 2\Big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu) \Big) \vec{A}^\nu = \vec{0}, \\
& Q_2 \vec{A}_\mu \equiv c_\alpha \partial_\alpha \vec{A}_\mu + 2\Big(w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu) \Big) \vec{A}^\nu = \vec{0}
\end{aligned}
$$

is invariant under the above mentioned group. Consequently, YMEs (7.1.1) are conditionally-invariant under the Lie algebra $\langle Q_1, Q_2, Q_3 \rangle$. It means that solutions of YMEs obtained with the help of Ansatz invariant under the group with generators Q_1, Q_2, Q_3 cannot be found by means of the classical symmetry reduction procedure.

As rather tedious computations show, the Ansätze determined by the formulae 2–4 from (7.2.6) also correspond to conditional symmetry of YMEs. Hence it follows, in particular, that YMEs should be included into the long list of mathematical and theoretical physics equations possessing nontrivial conditional symmetry [97].

THE POINCARÉ GROUP AND ITS REPRESENTATIONS

The Poincaré group $P(1,3)$ is a group of linear transformations of the Minkowski space $R(1,3)$ preserving the quadratic form $s(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. We say that there is a representation of the group $P(1,3)$ in some linear space H if the homomorphism of this group $g \to T_g$ into the set of linear operators on H is determined, i.e., the product of the group corresponds to the product of operators $T_{g_1 g_2} = T_{g_1} T_{g_2}$ and the unit element of the group $P(1,3)$ corresponds to the identical transformation of the space H. If the representation space H is infinite-dimensional, then it is assumed that the domain of definition of operators $T_g,\ \forall\, g \in P(1,3)$ is dense in H.

A representation is called irreducible if H contains no subspace invariant with respect to operators $T_g,\ \forall\, g \in P(1,3)$.

Irreducible representations of the Poincaré group were described by Wigner as early as 1939. It is known that the problem of description of representations of the Lie group G can be reduced to description of representations of its Lie algebra AG. An abstract definition of the algebra $AP(1,3)$ is given by commutation relations for the basis elements $P_\mu,\ J_{\alpha\beta}$

$$\begin{aligned} &[P_\mu,\, P_\nu] = 0, \quad [P_\mu,\, J_{\alpha\beta}] = i(g_{\mu\alpha} P_\nu - g_{\mu\beta} P_\mu), \\ &[J_{\mu\nu},\, J_{\alpha\beta}] = i(g_{\mu\beta} J_{\nu\alpha} + g_{\nu\alpha} J_{\mu\beta} - g_{\nu\beta} J_{\mu\alpha} - g_{\mu\alpha} J_{\nu\beta}). \end{aligned} \tag{A.1.1}$$

A homomorphism $x \to T(x)$ of the algebra $AP(1,3)$ into the set of linear operators determined in some linear space H

$$\begin{aligned} &ax + by \to aT(x) + bT(y), \\ &[x, y] \to [T(x), T(y)] = T(x)T(y) - T(y)T(x), \\ &\{x, y\} \subset AP(1,3), \quad \{a, b\} \subset \mathbb{C}^1 \end{aligned} \tag{A.1.2}$$

is called a representation of the Poincaré algebra $AP(1,3)$.

Wigner's results were supplemented by Shirokov [258] who was the first to construct an explicit form of the basis elements of the algebra $AP(1,3)$ for all classes of irreducible representations. In many successive papers representations of this algebra in various bases were found (see, for example, [28]).

We adduce the formulae giving a complete description of irreducible representations of the Poincaré algebra in the class of Hermitian operators following [115, 116, 118].

According to the Schur's lemma classification of irreducible representations of the Lie algebra L is reduced to construction of the complete set of operators commuting with all basis elements $x \in L$ (such operators are called the Casimir operators of the algebra L) and to computation of the spectrum of their eigenvalues. Furthermore, each set of eigenvalues of all Casimir operators corresponds to the one and only one irreducible representation [19].

Theorem A.1.1[118]. *An arbitrary Hermitian representation of the Poincaré algebra $AP(1,3)$ can be realized by the following operators:*

$$\begin{aligned} P_0 &= p_0, \quad P_a = p_a, \\ \vec{J} &= \vec{x}\times\vec{p} + \lambda_0 \frac{(\vec{n}+\vec{p})}{(1+\vec{n}\,\vec{p})}, \\ \vec{N} &= -p_0\vec{x} + \frac{\vec{\lambda}\times\vec{p}}{p^2} - (\lambda_0 p_0 p - \vec{\lambda}\,\vec{p})\frac{(\vec{p}\times\vec{n}\,)}{(p+\vec{n}\,\vec{p})}, \end{aligned} \tag{A.1.3}$$

where $\vec{J} = (J_1, J_2, J_3)$, $\vec{N} = (N_1, N_2, N_3)$, $J_a = (1/2)\varepsilon_{abc}J_{bc}$, $N_a = J_{0a}$, $a = 1, 2, 3$, p_0, p_a are real variables connected by the relation $p_0 = \varepsilon(C_1 + p_a p_a)^{1/2}$, $\varepsilon = \pm 1$, C_1 is an arbitrary real number, $x_a = i\partial/\partial p_a$, $a = 1, 2, 3$, $p = (p_a p_a)^{1/2}$; λ_0, λ_1, λ_2, λ_3 are matrices satisfying the commutation relations

$$\begin{aligned} [\lambda_0,\, \lambda_a] &= i\varepsilon_{abc} n_b \lambda_c, \\ [\lambda_a,\, \lambda_b] &= iC_1 \varepsilon_{abc} n_c \lambda_0 \end{aligned} \tag{A.1.4}$$

and $\vec{n} = (n_1, n_2, n_3)$ is an arbitrary unit vector.

The algebra (A.1.4) has two Casimir operators

$$I_1 = \lambda_0^2 C_1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \exp\{2i\pi\lambda_0\}.$$

To obtain the explicit forms of matrices λ_μ realizing an irreducible representation of algebra (A.1.4) we choose the basis which consists of the complete

set of eigenvectors of the commuting operators $I_1,\ I_2$ and λ_0. On designating these vectors by the symbol $|C_1, C_2, \lambda\rangle$ we have

$$
\begin{aligned}
&\lambda_0|C_1, C_2, \lambda\rangle = \lambda|C_1, C_2, \lambda\rangle,\\
&(\lambda_1 \pm i\lambda_2)|C_1, C_2, \lambda\rangle = (1/2)(1+n_3)\Big(C_2 - C_1\lambda(\lambda \pm 1)\Big)^{1/2}\\
&\times|C_1, C_2, \lambda \pm 1\rangle + (n_2 \mp n_1)^2\Big(2(1+n_3)\Big)^{-1}\\
&\times\Big(C_2 - C_1\lambda(\lambda \mp 1)\Big)^{1/2}|C_1, C_2, \lambda \mp 1\rangle,\\
&\lambda_3|C_1, C_2, \lambda\rangle = -(\lambda_1 + \lambda_2)|C_1, C_2, \lambda\rangle.
\end{aligned}
\tag{A.1.5}
$$

If the representation is irreducible, then the parameters $C_1,\ C_2$ take the fixed values from the intervals enumerated below in formulae (A.1.6)

$$
\begin{aligned}
&1)\quad C_1 = m^2 > 0, \quad C_2 = C_1 s(s+1), \quad \lambda = -s,\ -s+1, \ldots, s;\\
&2)\quad C_1 = C_2 = 0, \quad \lambda = \tilde{\lambda};\\
&3)\quad C_1 = 0, \quad C_2 = \eta^2 > 0,\\
&\qquad \lambda = 0,\ \pm 1,\ \pm 2,\ \ldots \quad \text{or} \quad \lambda = \pm 1/2,\ \pm 3/2, \ldots;\\
&4)\quad C_1 = -\eta^2 < 0, \quad C_2 = -\alpha\eta^2, \quad -\infty < \alpha < -1/4,\\
&\qquad \lambda = \pm 1/2,\ \pm 3/2, \ldots \quad \text{or} \quad \lambda = 0,\ \pm 1,\ \pm 2, \ldots;\\
&\qquad C_1 = -\eta^2 < 0, \quad 0 < C_2 < (1/4)\eta^2, \quad \lambda = 0, \pm 1, \pm 2, \ldots;\\
&\qquad C_1 = -\eta^2 < 0, \quad C_2 = -l(l+1)\eta^2, \quad \lambda = l+1,\ l+2, \ldots;\\
&\qquad C_1 = -\eta^2 < 0, \quad C_2 = -l(l+1)\eta^2, \quad \lambda = -l-1,\ -l-2, \ldots,
\end{aligned}
\tag{A.1.6}
$$

where $s > 0$ and $\tilde{\lambda}$ are arbitrary integer or half-integer numbers, l is a positive integer or half-integer number satisfying the condition $-(1/2) \le l < +\infty$ whose values in the irreducible representation are fixed.

Formulae (A.1.3)–(A.1.6) give all possible (up to the equivalence relation) irreducible Hermitian representations of the commutation relations (A.1.1) provided not all P_μ vanish. If $P_\mu = 0,\ \mu = 0, \ldots, 3$, then algebra (1.1.31) is isomorphic to the Lie algebra of the Lorentz group $O(1,3)$. The theory of representations of the algebra $AO(1,3)$ is expounded with exhaustive completeness in [174].

Among all possible representations of the Poincaré algebra a specific role is played by so-called covariant representations which are characterized by the following form of the basis elements

$$
P_\mu = p_\mu = ig_{\mu\nu}\partial_\nu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{A.1.7}
$$

where $S_{\mu\nu}$ are constant matrices.

Necessary and sufficient conditions for operators (A.1.7) to realize a representation of the Poincaré algebra are as follows:

$$[S_{\mu\nu},\, S_{\alpha\beta}] = i(g_{\mu\beta}S_{\nu\alpha} + g_{\nu\alpha}S_{\mu\beta} - g_{\mu\alpha}S_{\nu\beta} - g_{\nu\beta}S_{\mu\alpha}).$$

Let us note that operators (A.1.7) unlike those given by (A.1.3) realize a reducible representation of the algebra $AP(1,3)$. In addition, this representation is non-Hermitian if the matrices $S_{\mu\nu}$ are finite-dimensional.

In what follows we consider the case of finite-dimensional matrices $S_{\mu\nu}$, since it is mostly used in applications.

It is straightforward to verify that the matrices

$$\begin{aligned} j_a &= (1/2)\Big((1/2)\varepsilon_{abc}S_{bc} + iS_{0a}\Big), \\ \tau_a &= (1/2)\Big((1/2)\varepsilon_{abc}S_{bc} - iS_{0a}\Big) \end{aligned}$$

satisfy the following commutation relations:

$$[j_a,\, j_b] = i\varepsilon_{abc}j_c, \quad [\tau_a,\, \tau_b] = i\varepsilon_{abc}\tau_c, \quad [j_a,\, \tau_b] = 0. \tag{A.1.8}$$

As a basis of the space of a finite-dimensional irreducible representation of algebra (A.1.8) we take the complete set of eigenvectors $|j,m;\tau,n\rangle$ of commuting operators $j_aj_a,\ j_3,\ \tau_a\tau_a,\ \tau_3$. In this basis the action of the operators j_a and τ_a can be represented in the form

$$\begin{aligned} & j_aj_a|j,m;\tau,n\rangle = j(j+1)|j,m;\tau,n\rangle, \\ & j_3|j,m;\tau,n\rangle = m|j,m;\tau,n\rangle, \\ & (j_1 \pm ij_2)|j,m;\tau,n\rangle = \Big(j(j+1) \\ & \quad -m(m\pm1)\Big)^{1/2}|j,m\pm1;\tau,n\rangle, \\ & \tau_a\tau_a|j,m;\tau,n\rangle = \tau(\tau+1)|j,m;\tau,n\rangle, \\ & \tau_3|j,m;\tau,n\rangle = n|j,m;\tau,n\rangle, \\ & (\tau_1 \pm i\tau_2)|j,m;\tau,n\rangle = \Big(\tau(\tau+1) \\ & \quad -n(n\pm1)\Big)^{1/2}|j,m;\tau,n\pm1\rangle, \end{aligned} \tag{A.1.9}$$

where $j,\ m\,(\tau,\ n)$ are (half-) integer numbers, inequalities holding

$$-j \le m \le j, \quad -\tau \le n \le \tau.$$

Thus, irreducible finite-dimensional representations of the algebra $AO(1,3)$ (A.1.8) are realized by matrices of the dimension $(2j+1)(2\tau+1)\times(2j+1)(2\tau+1)$ with matrix elements (A.1.9). The above representations are denoted by the symbol $D(j,\tau)$.

Using formulae (A.1.9) it is easy to check that on the set of solutions of the Dirac equation (1.1.1) the representation $D(1/2,0)\oplus D(0,1/2)$ is realized.

The Poincaré algebra has two principal Casimir operators

$$I_1 = P_\mu P^\mu, \quad I_2 = W_\mu W^\mu,$$

where $W_0 = (1/2)\varepsilon_{abc}P_a J_{bc}$, $W_a = (1/2)P_0\varepsilon_{abc}J_{bc} - \varepsilon_{abc}P_b J_{0c}$, whose eigenvalues are considered as the mass and the spin of a particle.

We say that the Poincaré-invariant equation describes a particle with the spin s and the mass m provided its solutions satisfy identically the relations

$$I_1\psi = m^2\psi, \quad I_2\psi = s(s+1)m^2\psi.$$

It is established by direct computation that solutions of the Dirac equation satisfy the equalities $I_1\psi = m^2\psi$, $I_2\psi = (3/4)m^2\psi$, whence it follows that the Dirac equation (1.1.1) describes a particle with the spin $s = 1/2$ and the mass m.

THE GALILEI GROUP AND ITS REPRESENTATIONS

The Galilei group $G(1,3)$ is a group of transformations of the four-dimensional space $\mathbb{R}^1 \times \mathbb{R}^3$ of the form

$$\begin{aligned} t' &= t + r_0, \\ x'_a &= \theta_{ab} x_b + v_a t + r_a, \end{aligned} \tag{A.2.1}$$

where $\|\theta_{ab}\|_{a,b=1}^3$ is an arbitrary orthogonal matrix, $v_a,\ r_\mu$ are real parameters.

Since elements of orthogonal (3×3)-matrix can be expressed via three parameters (for example, via the Euler angles), the group (A.2.1) is a 10-parameter Lie transformation group.

It is worth noting that a condition of invariance of physical laws with respect to coordinate transformation (A.2.1) is nothing else but the mathematical formulation of the Galilei relativity principle. This principle establishes an equivalence of inertial reference frames. Therefore, the corresponding motion equation has to be invariant under the Galilei group. In other words, some representation of the Galilei group is to be realized on the set of solutions of the equation in question. Consequently, to investigate wave equations invariant under the group $G(1,3)$ we have to know its representations.

As noted in the Appendix 1 the problem of description of representations of the Lie group reduces to the study of representations of its Lie algebra and besides we can restrict ourselves to irreducible representations.

An abstract definition of the Galilei algebra $AG(1,3)$ with basis operators $P_0,\ P_a,\ J_a,\ G_a,\ M$ is given by the following commutation relations

$$[P_\mu,\, P_\nu] = 0, \quad [P_\mu,\, M] = 0,$$
$$[J_a,\, M] = 0, \quad [G_a,\, M] = 0,$$

$$[P_0,\, G_a] = iP_a, \quad [P_0,\, J_a] = 0, \qquad \text{(A.2.2)}$$
$$[P_a,\, G_b] = \delta_{ab} M, \quad [P_a,\, J_b] = i\varepsilon_{abc} P_c,$$
$$[G_a,\, J_b] = i\varepsilon_{abc} G_c, \quad [J_a,\, J_b] = i\varepsilon_{abc} J_c$$

where $\mu, \nu = 0, 1, 2, 3;\ a, b, c = 1, 2, 3.$

Note A.2.1. In Section 4.1 we designate the basis elements of the rotation group $J_a,\ a = 1, 2, 3$ as $J_{ab},\ a \neq b,\ a, b = 1, 2, 3.$ These notations are related by means of the formula

$$J_a = (1/2)\varepsilon_{abc} J_{bc}.$$

Let us note that the Lie algebra of the group (A.2.1) satisfies relations (A.2.2) under $M = 0$.

The algebra (A.2.2) has three principal Casimir operators

$$\begin{aligned} C_1 &= M, \\ C_2 &= (M\vec{J} - \vec{P} \times \vec{G})^2, \\ C_3 &= 2MP_0 - P_a P_a. \end{aligned} \qquad \text{(A.2.3)}$$

Following [114, 118] we give a realization of irreducible representations of the Galilei algebra distinguished by a universal and quite simple form of the generators of the group $G(1, 3)$.

Theorem A.2.1. *Irreducible Hermitian representations of the Galilei algebra $AG(1, 3)$ are numbered by numbers $C_1,\ C_2,\ C_3$ (eigenvalues of the Casimir operators (A.2.3)) which take the values*

$$\begin{aligned} &1)\quad C_1^2 = m^2 > 0, \quad C_2 = m^2 s(s+1), \quad -\infty < C_3 < +\infty, \\ &\qquad s = 0,\, 1/2,\, 1, \ldots; \\ &2)\quad C_1 = C_2 = 0, \quad C_3 = -k^2 < 0; \\ &3)\quad C_1 = 0, \quad C_2 = r^2, \quad C_3 = -k^2 < 0. \end{aligned} \qquad \text{(A.2.4)}$$

The explicit form of basis operators of an irreducible representation is determined by the formulae

$$P_0 = p_0, \quad P_a = p_a, \qquad \text{(A.2.5)}$$
$$M = C_1 = m,$$
$$J_a = -i\varepsilon_{abc} p_b \frac{\partial}{\partial p_c} + \lambda_0 \frac{(p_a + n_a)}{(1 + \vec{n}\,\vec{p})}, \qquad \text{(A.2.6)}$$

$$
\begin{aligned}
G_a \quad = \quad & -ip_a\frac{\partial}{\partial p_0} - im\frac{\partial}{\partial p_a} + \varepsilon_{abc}\frac{\lambda_b p_c}{(\vec{p}\,\vec{p})} \\
& -\varepsilon_{abc}p_b n_c\frac{(m\lambda_0 - \vec{\lambda}\,\vec{p})}{(p + \vec{n}\,\vec{p})},
\end{aligned}
$$

where m is a fixed real number, λ_μ are matrices (A.1.4)–(A.1.6) and the variables $p_0,\ p_a$ are connected by the relation

$$2mp_0 - p_a p_a = C_3,$$

C_3 being fixed too.

Let us give a brief characterization of the classes of irreducible representations enumerated in (A.2.4):

1) representations of the class I $(m \neq 0,\ m^2 > 0)$ are characterized by three numbers $m,\ s$ and ε_0, where m and ε_0 are arbitrary real numbers, s is an integer or half-integer non-negative number. Such representations are realized in the space of square-integrable functions $f(\vec{p},\,\lambda)$, where

$$\lambda = -s,\ -s+1, \ldots,\ s,$$

i.e., the dimension of $f(\vec{p},\,\lambda)$ with respect to the index λ is equal to $2s+1$. The space of irreducible representation of the algebra $AG(1,3)$ is usually associated with the position space of a free particle having the mass m, the spin s and the internal energy $\varepsilon_0/2m$;

2) representations of the class II are given by the pair of numbers

$$C_3 < 0 \text{ and } C_4 = 0,\ 1/2,\ 1, \ldots.$$

These representations are one-dimensional and are realized in the space of square-integrable functions $g(p_0,\,\vec{p})$.

Representations of the Galilei algebra of the class II are realized on the set of solutions of equations describing fields with the zero rest mass, for example, Galilei-invariant electro-magnetic field [212, 213];

3) representations of the class III are numbered by the pair of positive numbers $r^2,\ k^2$. These representations are realized in the space of square-integrable functions $h(p_0,\,\vec{p},\,\lambda)$, where λ takes the infinite number of values

$$0,\ \pm 1,\ \pm 2, \ldots \text{ or } \pm 1/2,\ \pm 3/2, \ldots.$$

So far representations of the Galilei algebra of the class III have no applications in physics.

The above considered classes of representations of the algebra $AG(1,3)$ exhaust all inequivalent non-Hermitian representations of this algebra if not all P_μ are equal to zero.

Provided

$$P_\mu = 0, \quad \mu = 0, \ldots, 3,$$

the Galilei algebra is isomorphic to the Lie algebra of the Euclid group $AE(3)$ which is determined by commutation relations (4.3.4). The problem of complete description of inequivalent irreducible representations of the Euclid algebra is reduced to a purely algebraic problem which cannot be solved by already known methods [118]. By the same reason, the problem of description of all inequivalent covariant representations of the algebra $AG(1,3)$ having the form (4.3.3) is not solved yet.

REPRESENTATIONS OF THE POINCARÉ AND GALILEI GROUPS BY LIE VECTOR FIELDS

Given a fixed representation of a Lie transformations group G, the problem of description of differential equations invariant under the group G is reduced with the help of the infinitesimal Lie method to integrating some over-determined linear system of PDEs (called determining equations). But to solve the problem of constructing *all* differential equations admitting the transformation group G whose representation is not fixed *a priori* one has

- to construct all inequivalent (in some sense) representations of the Lie transformation group G,
- to solve the determining equations for each representation obtained.

And what is more, the first problem, in contrast to the second one, reduces to solving *nonlinear* systems of PDEs. It has been completely solved by Rideau and Winternitz [247], Zhdanov and Fushchych [307] for the generalized Galilei group $G_2(1,1)$ acting in the space of two dependent and two independent variables.

Some new representations of the Galilei group $G(1,3)$ were suggested in [102]–[104],[144]. Yehorchenko [288] and Fushchych, Tsyfra and Boyko [144] have constructed new (nonlinear) representations of the Poincaré groups $P(1,2)$ and $P(1,3)$, correspondingly. A complete description of *covariant* representations of the conformal group $C(n,m)$ in the space of $n+m$ independent and one dependent variables was obtained by Fushchych, Zhdanov and Lahno [110, 164]. It has been established, in particular, that any covariant representation of the Poincaré group $P(n,m)$ with $\max\{n,m\} \geq 3$ in the case of one

dependent variable is equivalent to the standard representation. But given the condition $\max\{n,m\} < 3$, there exist essentially new representations of the corresponding Poincaré groups.

In this appendix we give a brief account of our latest results on classification of inequivalent representation of the Euclid group $E(3)$, which is a semi-direct product of the three-parameter rotation group $O(3)$ and of the three-parameter Abelian group of translations $T(3)$, acting in the space of three independent x_1, x_2, x_3 and $n \in \mathbb{N}$ dependent $u_1, \dots, u_n$ variables. Furthermore, we adduce results on classification of representations of the Poincaré and Galilei groups acting in the space of four independent x_0, x_1, x_2, x_3 and $n \in \mathbb{N}$ dependent $u_1, \dots, u_n$ variables.

It is a common knowledge that investigation of representations of a Lie transformation group G is reduced to study of representations of its Lie algebra AG whose basis elements are first-order differential operators (Lie vector fields) of the form

$$Q = \xi_\alpha(x,u)\partial_{x_\alpha} + \eta_i(x,u)\partial_{u_i}, \tag{A.3.1}$$

where $\xi_\alpha,\ \eta_a$ are some real-valued smooth functions on $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$, $\partial_{x_\alpha} = \frac{\partial}{\partial x_\alpha}$, $\partial_{u_i} = \frac{\partial}{\partial u_i}$, $\alpha = 0, \dots, 3$, $i = 1, 2, \dots, n$.

In the above formulae we have two kinds of variables. The variables $x_0, \dots, x_3$ and $u_1, u_2, \dots, u_n$ will be referred to as independent and dependent variables, respectively. Difference between these becomes essential when we take into consideration partial differential equations invariant under the Lie algebra AG.

Due to the properties of the corresponding Lie transformation group G basis operators $Q_a,\ a = 1, \dots, N$ of the Lie algebra AG satisfy commutation relations

$$[Q_a,\ Q_b] = C_{ab}^c Q_c, \quad a, b = 1, \dots, N, \tag{A.3.2}$$

where $[Q_a,\ Q_b] \equiv Q_a Q_b - Q_b Q_a$ is the commutator.

In (A.3.2) $C_{ab}^c \in \mathbb{R}$ are structure constants which determine uniquely the Lie algebra AG. A fixed set of the Lie vector fields Q_a satisfying (A.3.2) is called the representation of the Lie algebra AG.

Thus, the problem of description of all representations of a given Lie algebra AG reduces to solving the relations (A.3.2) with some fixed structure constants C_{ab}^c in the class of Lie vector fields (A.3.1).

It is easy to check that the relations (A.3.2) are not altered with an arbi-

trary invertible transformation of variables $x,\ u$

$$\begin{aligned} y_\alpha &= f_\alpha(x,u), \qquad \alpha = 0,\dots,3, \\ v_i &= g_i(x,u), \qquad i = 1,\dots,n, \end{aligned} \tag{A.3.3}$$

where $f_\alpha,\ g_i$ are smooth functions. That is why, one can introduce on a set of representations of a Lie algebra AG the following relation: two representations $Q_1,\dots,Q_N$ and $Q'_1,\dots,Q'_N$ are called equivalent if they are transformed one into another by means of an invertible transformation (A.3.3). Since invertible transformations of the form (A.3.2) form the group (called diffeomorphism group), the relation above is the equivalence relation. It divides the set of all representations of the Lie algebra AG into equivalence classes $A_1,\dots,A_r$. Consequently, to describe all possible representations of AG it suffices to construct one representative of each equivalence class $A_j,\ j = 1,\dots,r$.

Definition A.3.1. Set of first-order linearly independent differential operators P_a, J_b of the form (A.3.1) is called the Euclid algebra $AE(3)$ if they satisfy the following commutation relations:

$$[P_a,\ P_b] = 0, \quad [J_a,\ P_b] = \varepsilon_{abc} P_c, \quad [J_a,\ J_b] = \varepsilon_{abc} J_c, \tag{A.3.4}$$

where

$$\varepsilon_{abc} = \begin{cases} 1, & (abc) = \text{cycle}\,(123), \\ -1, & (abc) = \text{cycle}\,(213), \\ 0, & \text{in the remaining cases.} \end{cases}$$

Definition A.3.2. Set of first-order linearly independent differential operators $P_\mu, J_{\alpha\beta}$ of the form (A.3.1) is called the Poincaré algebra $AP(1,3)$ if they satisfy the following commutation relations:

$$\begin{aligned} &[P_\mu,\ P_\nu] = 0, \quad [P_\mu,\ J_{\alpha\beta}] = g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha, \\ &[J_{\mu\nu},\ J_{\alpha\beta}] = g_{\mu\beta} J_{\nu\alpha} + g_{\nu\alpha} J_{\mu\beta} - g_{\mu\alpha} J_{\nu\beta} - g_{\nu\beta} J_{\mu\alpha}. \end{aligned} \tag{A.3.5}$$

Definition A.3.3. Set of first-order linearly independent differential operators P_0, P_a, J_b, G_c, M of the form (A.3.1) is called the Galilei algebra $AG(1,3)$ if they satisfy the commutation relations (A.3.4) and

$$\begin{aligned} &[P_0,\ P_a] = 0, \quad [P_0,\ J_a] = 0, \quad [P_0,\ G_a] = P_a, \\ &[P_0,\ M] = 0, \quad [P_a,\ G_b] = \delta_{ab} M, \quad [P_a,\ M] = 0, \\ &[J_a,\ G_b] = \varepsilon_{abc} G_c, \quad [G_a,\ G_b] = 0, \quad [G_a,\ M] = 0. \end{aligned} \tag{A.3.6}$$

We say that basis elements of the Euclid algebra $AE(3)$ realize a covariant representation if they can be reduced to the form

$$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + \eta_{ai}(x,u)\partial_{u_i} \tag{A.3.7}$$

with the help of the transformation (A.3.3).

Note that the case when η_{ai} are linear in u corresponds to what is called in the classical representation theory a covariant representation of the Euclid algebra (see Appendix 2). This is the reason why we preserve for the more general class of representations (A.3.7) the term 'covariant representation'.

Similarly, operators $P_\mu, J_{\alpha\beta}$ realize a covariant representation of the Poincaré algebra $AP(1,3)$ if they can be reduced to the form

$$P_\mu = g_{\mu\nu}\partial_{x_\nu}, \quad J_{\alpha\beta} = x_\alpha g_{\beta\nu}\partial_{x_\nu} - x_\beta g_{\alpha\nu}\partial_{x_\nu} + \eta_{\alpha\beta i}(x,u)\partial_{u_i} \tag{A.3.8}$$

with the help of a transformation (A.3.3).

At last, operators P_0, P_a, J_a, G_a, M realize a covariant representation of the Galilei algebra $AG(1,3)$ if they can be reduced to the form

$$\begin{aligned} &P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + \eta^1_{ai}(x,u)\partial_{u_i}, \\ &G_a = x_0 \partial_{x_a} + \eta^2_{ai}(x,u)\partial_{u_i}, \quad M = \eta^3_i(x,u)\partial_{u_i} \end{aligned} \tag{A.3.9}$$

with the help of a transformation (A.3.3).

A specific role played by covariant representations of the algebras $AE(3)$, $AP(1,3)$ and $AG(1,3)$ is explained by the fact that they are widely used in physical applications. Furthermore, the transformation groups generated by their basis elements have a natural physical interpretation. The operators P_0, P_a generate translations of the time x_0 and space x_a variables, correspondingly. Next, the operators J_a generate rotations of the Euclid space $\vec{x}$ and the operators J_{0a} generate the Lorentz transformations of the Minkowski space $x_0, \vec{x}$ preserving the quadratic form $x_\mu x^\mu$. The operators G_a generate the Galilei transformations of the space of independent variables x_0, x_a leaving the time variable x_0 invariant.

In what follows, we will restrict our considerations to the case of covariant representations only.

1. Covariant representations of the Euclid algebra. Direct check shows that the operators (A.3.7) form a basis of the Euclid algebra iff the following relations hold:

$$\frac{\partial \eta_{ai}}{\partial x_b} = 0, \quad [\eta_{ai}\partial_{u_i},\ \eta_{bj}\partial_{u_j}] = \varepsilon_{abc}\eta_{ci}\partial_{u_i},$$

where $a, b = 1, 2, 3,\ i = 1, \ldots, n$.

Consequently, functions η_{ai} are independent of x and, in addition, the operators

$$\mathcal{J}_a = \eta_{ai}(u)\partial_{u_i} \tag{A.3.10}$$

satisfy the commutation relations of the Lie algebra of the rotation group

$$[\mathcal{J}_a,\ \mathcal{J}_a] = \varepsilon_{abc}\mathcal{J}_c. \tag{A.3.11}$$

Thus, the problem of description of all inequivalent covariant representations reduces to describing all functions $\eta_{ai}(u)$ such that the operators $\mathcal{J}_a$ fulfill the commutation relations (A.3.11). Solution of this problem is given by the following lemma.

Lemma A.3.1. *Let the differential operators (A.3.10) satisfy the commutation relations (A.3.11). Then, there exists a transformation*

$$v_i = F_i(u), \quad i = 1, \ldots, n \tag{A.3.12}$$

reducing these operators to one of the following forms:

$$\begin{aligned} 1.\quad J_1 &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ J_2 &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ J_3 &= \partial_{u_1}; \end{aligned} \tag{A.3.13}$$

$$\begin{aligned} 2.\quad J_1 &= -\sin u_1 \tan u_2 \partial_{u_1} - (\cos u_1 - \alpha \sin u_1 \sec u_2)\partial_{u_2} \\ &\quad + \sin u_1 \sec u_2 \partial_{u_3}, \\ J_2 &= -\cos u_1 \tan u_2 \partial_{u_1} + (\sin u_1 + \alpha \cos u_1 \sec u_2)\partial_{u_2} \\ &\quad + \cos u_1 \sec u_2 \partial_{u_3}, \\ J_3 &= \partial_{u_1}; \end{aligned} \tag{A.3.14}$$

$$3.\quad J_a = 0, \quad a = 1, 2, 3. \tag{A.3.15}$$

Here α is an arbitrary smooth function of $u_3, \ldots, u_n$.

Proof. If at least one of the operators J_a (say J_3) is equal to zero, then by virtue of commutation relations (A.3.11) two other operators $J_2,\ J_3$ are also equal to zero and we get (A.3.15).

Let J_3 be a non-zero operator. Then, using a transformation (A.3.12) we can always reduce the operator J_3 to the form $J_3 = \partial_{v_1}$ (we should write J_3' but to simplify the notations we omit hereafter primes). Next, from the

commutation relations $[J_3,\ J_1] = J_2$, $[J_3,\ J_2] = -J_1$ it follows that coefficients of the operators $J_1,\ J_2$ satisfy the system of ordinary differential equations with respect to v_1

$$\frac{\partial \eta_{2i}}{\partial v_1} = \eta_{3i}, \quad \frac{\partial \eta_{3i}}{\partial v_1} = -\eta_{2i}. \quad i = 1, \ldots, n.$$

Solving the above system yields

$$\begin{aligned} \eta_{2i} &= f_i \cos v_1 + g_i \sin v_1, \\ \eta_{3i} &= g_i \cos v_1 - f_i \sin v_1, \end{aligned} \tag{A.3.16}$$

where $f_i,\ g_i$ are arbitrary smooth functions of $v_2, \ldots, v_n$, $i = 1, \ldots, n$.

Case 1. $f_j = g_j = 0, \quad j \geq 2.$

In this case operators $J_1,\ J_2$ read

$$J_1 = f \cos v_1 \partial_{v_1}, \quad J_2 = -f \sin v_1 \partial_{v_1}$$

with an arbitrary smooth function $f = f(v_2, \ldots, v_n)$.

Inserting the above expressions into the remaining commutation relation $[J_1,\ J_2] = J_3$ and computing the commutator on the left-hand side we arrive at the equality $f^2 = -1$ which can not be satisfied by a real-valued function.

Case 2. Not all $f_j,\ g_j,\ j \geq 2$ are equal to 0.

Making the change of variables

$$w_1 = v_1 + V(v_2, \ldots, v_n), \quad w_j = v_j, \quad j = 2, \ldots, n$$

we transform operators $J_a,\ a = 1, 2, 3$ with coefficients (A.3.16) as follows

$$\begin{aligned} J_1 &= \tilde{f} \sin w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{f}_j \cos w_1 + \tilde{g}_j \sin w_1) \partial_{w_j}, \\ J_2 &= \tilde{f} \cos w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{g}_j \cos w_1 - \tilde{f}_j \sin w_1) \partial_{w_j}, \\ J_3 &= \partial_{w_1}. \end{aligned} \tag{A.3.17}$$

Subcase 2.1. Not all $\tilde{f}_j$ are equal to 0. Making the transformation

$$z_1 = w_1, \quad z_j = W_j(w_2, \ldots, w_n), \quad j = 2, \ldots, n,$$

where W_2 is a particular solution of the PDE

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W_2 = 1,$$

and $W_3, \ldots, W_n$ are functionally-independent first integrals of the PDE

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W = 0$$

we reduce the operators (A.3.17) to be

$$\begin{aligned}
J_1 &= F \sin z_1 \partial_{z_1} + \cos z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \sin z_1 \partial_{z_j}, \\
J_2 &= F \cos z_1 \partial_{z_1} - \sin z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \cos z_1 \partial_{w_j}, \qquad \text{(A.3.18)} \\
J_3 &= \partial_{z_1}.
\end{aligned}$$

Substituting operators (A.3.18) into the commutation relation $[J_1,\ J_2] = J_3$ and equating coefficients of the linearly independent operators $\partial_{z_1}, \ldots, \partial_{z_n}$ we arrive at the following system of PDEs for the functions $F,\ G_2, \ldots, G_n$:

$$F_{z_2} - F^2 = 1, \quad G_{jz_2} - FG_j = 0, \quad j = 2, \ldots, n.$$

Integration of the above equations yields

$$\begin{aligned}
F &= \tan(z_2 + c_1), \\
G_j &= \frac{c_j}{\cos(z_2 + c_1)},
\end{aligned}$$

where $c_1, \ldots, c_n$ are arbitrary smooth functions of $z_3, \ldots, z_n$, $j = 2, \ldots, n$.

Replacing, if necessary, z_2 by $z_2 + c_1(z_3, \ldots, z_n)$ we may put c_1 equal to zero. Next, making the transformation

$$\begin{aligned}
y_a &= z_a, \quad a = 1, 2, 3, \\
y_k &= Z_k(z_3, \ldots, z_n), \quad k = 4, \ldots, n,
\end{aligned}$$

where Z_k are functionally-independent first integrals of the PDE

$$\sum_{j=3}^{n} G_j \partial_{z_j} Z = 0,$$

we can put $G_k = 0,\ k = 4, \dots, n$.

With these remarks the operators (A.3.18) take the form

$$\begin{aligned} J_1 &= \sin y_1 \tan y_2 \partial_{y_1} + \cos y_1 \partial_{y_2} + \frac{\sin y_1}{\cos y_2}(f\partial_{y_2} + g\partial_{y_3}), \\ J_2 &= \cos y_1 \tan y_2 \partial_{y_1} - \sin y_1 \partial_{y_2} + \frac{\cos y_1}{\cos y_2}(f\partial_{y_2} + g\partial_{y_3}), \\ J_3 &= \partial_{y_1}, \end{aligned} \quad \text{(A.3.19)}$$

where $f,\ g$ are arbitrary smooth functions of $y_3, \dots, y_n$.

If in (A.3.19) $g \not\equiv 0$, then replacing y_3 by $\tilde{y}_3 = \int g^{-1} dy_3$ and y_2 by $\tilde{y}_2 = -y_2$ we transform the above operators to the form (A.3.14).

If $g \equiv 0$, then making the transformation

$$\tilde{u}_1 = y_1 + \arctan \frac{f}{\cos y_2}, \quad \tilde{u}_2 = -\arctan \frac{\sin y_2}{\sqrt{\cos^2 y_2 + f^2}}, \quad \tilde{u}_k = y_k,$$

where $k = 3, \dots, n$, we reduce the operators (A.3.19) to the form (A.3.13).

Subcase 2.2. $f_j = 0,\ j = 2, \dots, n$.

Substituting operators (A.3.17) under $f_j = 0$ into the commutation relation $[J_1,\ J_2] = J_3$ and equating coefficients of the linearly independent operators $\partial_{z_1}, \dots, \partial_{z_n}$ yield the following system of PDEs:

$$-f^2 = 1, \quad fg_j = 0, \quad j = 2, \dots, n.$$

As the function f is real-valued, the system obtained is inconsistent.

Thus, we have proved that operators (A.3.12)–(A.3.15) exhaust a set of all possible inequivalent representations of the Lie algebra with commutation relations (A.3.11) in the class of the first-order differential operators (A.3.10).

As an immediate consequence of Lemma A.3.1 we get the following assertion.

Theorem A.3.1. *Any covariant representation of the Euclid algebra is equivalent to one of the following representations:*

1. $P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c};$ (A.3.20)

2. $$\begin{aligned} &P_a = \partial_{x_a}, \\ &J_1 = x_3\partial_{x_2} - x_2\partial_{x_3} - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ &J_2 = x_1\partial_{x_3} - x_3\partial_{x_1} - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ &J_3 = x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_{u_1}; \end{aligned} \quad \text{(A.3.21)}$$

$$
\begin{aligned}
3.\quad & P_a = \partial_{x_a}, \\
& J_1 = x_3\partial_{x_2} - x_2\partial_{x_3} - \sin u_1 \tan u_2 \partial_{u_1} \\
& \qquad - (\cos u_1 - \alpha \sin u_1 \sec u_2)\partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\
& J_2 = x_1\partial_{x_3} - x_3\partial_{x_1} - \cos u_1 \tan u_2 \partial_{u_1} \\
& \qquad + (\sin u_1 + \alpha \cos u_1 \sec u_2)\partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\
& J_3 = x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_{u_1}.
\end{aligned}
\tag{A.3.22}
$$

Here α *is an arbitrary smooth function of* $u_3, \ldots, u_n$*.*

In two next subsections we will give without proofs the assertions describing inequivalent covariant representations of the Poincaré and Galilei algebras.

2. Covariant representations of the Poincaré algebra. Inserting the operators (A.3.8) into commutation relations (A.3.5) yields that the functions $\eta_{\alpha\beta i}(x,u)$ are independent of x and the operators

$$
\mathcal{J}_{\alpha\beta} = \eta_{\alpha\beta i}(u)\partial_{u_i} \tag{A.3.23}
$$

satisfy the commutation relations of the Lie algebra of the Lorentz group

$$
[\mathcal{J}_{\mu\nu},\ \mathcal{J}_{\alpha\beta}] = g_{\mu\beta}\mathcal{J}_{\nu\alpha} + g_{\nu\alpha}\mathcal{J}_{\mu\beta} - g_{\mu\alpha}\mathcal{J}_{\nu\beta} - g_{\nu\beta}\mathcal{J}_{\mu\alpha}.
$$

Consequently, the problem of describing inequivalent covariant representations of the Poincaré algebra reduces to describing inequivalent representations of the Lorentz algebra having the basis elements (A.3.23).

Theorem A.3.2. *Any covariant representation of the Poincaré algebra is equivalent to the representation*

$$
\begin{aligned}
& P_\mu = g_{\mu\nu}\partial_{x_\nu}, \\
& J_{0i} = -x_0\partial_{x_i} - x_i\partial_{x_0} + \frac{1}{2}(\mathcal{P}_i + \mathcal{K}_i), \\
& J_{i3} = x_3\partial_{x_i} - x_i\partial_{x_3} + \frac{1}{2}(\mathcal{P}_i - \mathcal{K}_i), \\
& J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2} + \mathcal{J}_{12}, \\
& J_{03} = -x_0\partial_{x_3} - x_3\partial_{x_0} + \mathcal{D},
\end{aligned}
$$

where $i = 1, 2$ *and the operators* $\mathcal{P}_i, \mathcal{J}_{12}, \mathcal{D}, \mathcal{K}_i$ *are given by one of the formulae below*

1. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$$\begin{aligned}
&\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2},\\
&\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2},\\
&\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2};
\end{aligned}$$

2. $$\begin{aligned}
&\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},\\
&\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_3},\\
&\mathcal{K}_1 = (-u_1^2 + u_2^2 + \epsilon e^{-2u_3})\partial_{u_1} - 2u_1u_2\partial_{u_2} + 2u_1\partial_{u_3},\\
&\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2 + \epsilon e^{-2u_3})\partial_{u_2} + 2u_2\partial_{u_3};
\end{aligned}$$

3. $$\begin{aligned}
&\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},\\
&\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2},\\
&\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + 2u_2\partial_{u_3},\\
&\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - 2u_1\partial_{u_3};
\end{aligned}$$

4. $$\begin{aligned}
&\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},\\
&\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},\\
&\mathcal{K}_1 = (-u_1^2 + u_2^2 + \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_1} - (2u_1u_2 + \epsilon e^{-2u_4}\sin 2u_3)\partial_{u_2}\\
&\qquad + \Big(2u_2 + (q\cos u_3 + r\sin u_3)e^{-u_4}\Big)\partial_{u_3} + \Big(2u_1 - (r\cos u_3\\
&\qquad - q\sin u_3)e^{-u_4}\Big)\partial_{u_4} + e^{-u_4}\sin u_3\partial_{u_5} + e^{-u_4}\cos u_3\partial_{u_6},\\
&\mathcal{K}_2 = (-2u_1u_2 - \epsilon e^{-2u_4}\sin 2u_3)\partial_{u_1} + (u_1^2 - u_2^2 - \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_2}\\
&\qquad - \Big(2u_1 + (q\sin u_3 - r\cos u_3)e^{-u_4}\Big)\partial_{u_3} + \Big(2u_2 + (r\sin u_3\\
&\qquad + q\cos u_3)e^{-u_4}\Big)\partial_{u_4} + e^{-u_4}\cos u_3\partial_{u_5} - e^{-u_4}\sin u_3\partial_{u_6};
\end{aligned}$$

5. $$\begin{aligned}
&\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},\\
&\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},\\
&\mathcal{K}_1 = (-u_1^2 + u_2^2 + \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_1} - (2u_1u_2 + \epsilon e^{-2u_4}\sin 2u_3)\partial_{u_2}\\
&\qquad + \Big(2u_2 + (f\cos u_3 + g\sin u_3)e^{-u_4}\Big)\partial_{u_3} + \Big(2u_1 - (g\cos u_3\\
&\qquad - f\sin u_3)e^{-u_4}\Big)\partial_{u_4} + (h\cos u_3 + \sin u_3)e^{-u_4}\partial_{u_5},\\
&\mathcal{K}_2 = -(2u_1u_2 + \epsilon e^{2u_4}\sin 2u_3)\partial_{u_1} + (u_1^2 - u_2^2 - \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_2}\\
&\qquad + \Big(-2u_1 + (g\cos u_3 - f\sin u_3)e^{-u_4}\Big)\partial_{u_3} + \Big(2u_2 + (f\cos u_3\\
&\qquad + g\sin u_3)e^{-u_4}\Big)\partial_{u_4} + (\cos u_3 - h\sin u_3)e^{-u_4}\partial_{u_5};
\end{aligned}$$

6. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},$

$\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + (2u_2 + \epsilon e^{-u_4}\cos u_3)\partial_{u_3}$
$\quad + \Big(2u_1 + (fe^{-\epsilon u_5}\cos u_3 + \epsilon\sin u_3)e^{-u_4}\Big)\partial_{u_4}$
$\quad + \Big(((\epsilon fe^{-\epsilon u_5} + g)\cos u_3 + \sin u_3)e^{-u_4}\Big)\partial_{u_5} + he^{-u_4}\cos u_3\partial_{u_6},$

$\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - (2u_1 + \epsilon e^{-u_4}\sin u_3)\partial_{u_3}$
$\quad + \Big(2u_2 + (\epsilon\cos u_3 - fe^{-\epsilon u_5}\sin u_3)e^{-u_4}\Big)\partial_{u_4}$
$\quad + \Big(\cos u_3 - (\epsilon fe^{-\epsilon u_5} + g)\sin u_3)e^{-u_4}\Big)\partial_{u_5} - he^{-u_4}\sin u_3\partial_{u_6};$

7. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},$

$\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + 2u_2\partial_{u_3} + (2u_1 + fe^{-u_4}\cos u_3)\partial_{u_4}$
$\quad + \Big((-u_5f + g)\cos u_3 + \sin u_3)e^{-u_4}\Big)\partial_{u_5} + he^{-u_4}\cos u_3\partial_{u_6},$

$\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - 2u_1\partial_{u_3} + (2u_2 - fe^{-u_4}\sin u_3)\partial_{u_4}$
$\quad + \Big((\cos u_3 + (u_5f - g)\sin u_3)e^{-u_4}\Big)\partial_{u_5} - he^{-u_4}\sin u_3\partial_{u_6};$

8. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},$

$\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + (2u_2 + \epsilon e^{-u_4}\cos u_3)\partial_{u_3}$
$\quad + (2u_1 + \epsilon e^{-u_4}\sin u_3)\partial_{u_4},$

$\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - (2u_1 + \epsilon e^{-u_4}\sin u_3)\partial_{u_3}$
$\quad + (2u_2 + \epsilon e^{-u_4}\cos u_3)\partial_{u_4};$

9. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$\mathcal{J}_{12} = u_2\partial_{u_2} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},$

$\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + 2u_2\partial_{u_3} + (2u_1 + \epsilon e^{-u_4}\sin u_3)\partial_{u_4},$

$\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - 2u_1\partial_{u_3} + (2u_2 + \epsilon e^{-u_4}\cos u_3)\partial_{u_4};$

10. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$

$\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + \partial_{u_4},$

$\mathcal{K}_1 = (-{u_1}^2 + {u_2}^2 + \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_1} - (2u_1u_2 + \epsilon e^{-2u_4}\sin 2u_3)\partial_{u_2}$

$$\begin{aligned}&+ 2u_2\partial_{u_3} + 2u_1\partial_{u_4},\\ \mathcal{K}_2 &= -(2u_1u_2 + \epsilon e^{-2u_4}\sin 2u_3)\partial_{u_1} + ({u_1}^2 - {u_2}^2 - \epsilon e^{-2u_4}\cos 2u_3)\partial_{u_2}\\ &- 2u_1\partial_{u_3} + 2u_2\partial_{u_4};\end{aligned}$$

11. $\mathcal{P}_1 = \partial_{u_1}, \quad \mathcal{P}_2 = \partial_{u_2},$
$\mathcal{J}_{12} = u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \quad \mathcal{D} = -u_1\partial_{u_1} - u_2\partial_{u_2} + Q\partial_{u_3},$
$\mathcal{K}_1 = (-u_1^2 + u_2^2)\partial_{u_1} - 2u_1u_2\partial_{u_2} + 2(u_2 + u_1Q)\partial_{u_3},$
$\mathcal{K}_2 = -2u_1u_2\partial_{u_1} + (u_1^2 - u_2^2)\partial_{u_2} - 2(u_1 - u_2Q)\partial_{u_3}.$

Here $\epsilon = 0, 1$, and f, g, h are arbitrary smooth functions of $u_6, \ldots, u_n$, and Q is an arbitrary smooth function of $u_4, \ldots, u_n$, and

$$\begin{aligned} r &= U(u_5 + iu_6, u_7, \ldots, u_n) + U(u_5 - iu_6, u_7, \ldots, u_n),\\ q &= i\Big(U(u_5 + iu_6, u_7, \ldots, u_n) - U(u_5 - iu_6, u_7, \ldots, u_n)\Big)\end{aligned}$$

with an arbitrary function U analytic in the variable $u_5 + iu_6$.

Note that the operators $\mathcal{P}_i, \mathcal{J}_{12}, \mathcal{D}, \mathcal{K}_i$ fulfill the commutation relations of the Lie algebra of the conformal group $C(2)$ (which is isomorphic to the Lorentz algebra $AO(1,3)$)

$$\begin{aligned}&[\mathcal{P}_i,\, \mathcal{D}] = -\mathcal{P}_i, \quad [\mathcal{P}_1,\, \mathcal{J}_{12}] = -\mathcal{P}_2, \quad [\mathcal{P}_2,\, \mathcal{J}_{12}] = \mathcal{P}_1,\\ &[\mathcal{J}_{12},\, \mathcal{D}] = 0, \quad [\mathcal{P}_1,\, \mathcal{K}_1] = [\mathcal{P}_2,\, \mathcal{K}_2] = \mathcal{D},\\ &[\mathcal{P}_1,\, \mathcal{K}_2] = -2\mathcal{J}_{12}, \quad [\mathcal{P}_2,\, \mathcal{K}_1] = 2\mathcal{J}_{12},\\ &[\mathcal{K}_i,\, \mathcal{D}] = \mathcal{K}_i, \quad [\mathcal{K}_1,\, \mathcal{J}_{12}] = -\mathcal{K}_2, \quad [\mathcal{K}_2,\, \mathcal{J}_{12}] = \mathcal{K}_1.\end{aligned}$$

The above formulae give the list of all inequivalent representations of the algebra $AC(2)$ by Lie vector fields.

3. Covariant representations of the Galilei algebra. Inserting the formulae (A.3.9) into (A.3.6) and making some simple manipulations we conclude that the basis elements of a covariant representation of the algebra $AG(1,3)$ necessarily take the form

$$\begin{aligned}&P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a}, \quad J_a = \varepsilon_{abc}x_c\partial_{x_b} + \mathcal{J}_a,\\ &G_a = x_0\partial_{x_a} + x_a\mathcal{M} + \mathcal{G}_a, \quad M = \mathcal{M},\end{aligned} \tag{A.3.24}$$

where $\mathcal{J}_a, \mathcal{G}_b, \mathcal{M}$ are Lie vector fields of the form $\eta_i(u)\partial_{u_i}$ satisfying the commutation relations of the Euclid algebra

$$[\mathcal{G}_a,\ \mathcal{G}_b] = 0, \quad [\mathcal{J}_a,\ \mathcal{G}_b] = \varepsilon_{abc}\mathcal{G}_c, \quad [\mathcal{J}_a,\ \mathcal{J}_b] = \varepsilon_{abc}\mathcal{J}_c$$

and

$$[\mathcal{M},\ \mathcal{J}_a] = 0, \quad [\mathcal{M},\ \mathcal{G}_a] = 0.$$

On describing all inequivalent representations of the above Lie algebra we arrive at the following assertion.

Theorem A.3.3. *Any covariant representation of the Galilei algebra $AG(1,3)$ is equivalent to the representation having the basis elements (A.3.24), operators $\mathcal{J}_a, \mathcal{G}_b, \mathcal{M}$ being given by one of the formulae below*

1. $$\mathcal{J}_1 = u_3\partial_{u_2} - u_2\partial_{u_3}, \quad \mathcal{J}_2 = u_1\partial_{u_3} - u_3\partial_{u_1}, \quad \mathcal{J}_3 = u_2\partial_{u_1} - u_1\partial_{u_2},$$
$$\mathcal{G}_1 = \partial_{u_1}, \quad \mathcal{G}_2 = \partial_{u_2}, \quad \mathcal{G}_3 = \partial_{u_3},$$
$$\mathcal{M} = \epsilon\partial_{u_4},$$

2. $$\begin{aligned}\mathcal{J}_1 &= -u_2 \cos u_3 \tan u_4 \partial_{u_1} + u_2 \sin u_3 \tan u_4 \partial_{u_2} + \cos u_3 \cot u_4 \partial_{u_3} \\ &\quad + \sin u_3 \partial_{u_4} + \cos u_3 \csc u_4 \partial_{u_5}, \\ \mathcal{J}_2 &= u_1 \cos u_3 \tan u_4 \partial_{u_1} - u_1 \sin u_3 \tan u_4 \partial_{u_2} - \cot u_4 \sin u_3 \partial_{u_3} \\ &\quad + \cos u_3 \partial_{u_4} - \csc u_4 \sin u_3 \partial_{u_5}, \\ \mathcal{J}_3 &= u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \\ \mathcal{G}_1 &= \partial_{u_1}, \quad \mathcal{G}_2 = \partial_{u_2}, \quad \mathcal{G}_3 = \cos u_3 \tan u_4 \partial_{u_1} - \sin u_3 \tan u_4 \partial_{u_2}, \\ \mathcal{M} &= f(\cos u_3 \cos u_5 \sec u_4 + \sin u_3 \sin u_5)\partial_{u_1} \\ &\quad + f(-\sin u_3 \sec u_4 \cos u_5 + \cos u_3 \sin u_5)\partial_{u_2} + g\partial_{u_5} + \epsilon\partial_{u_6};\end{aligned}$$

3. $$\begin{aligned}\mathcal{J}_1 &= -u_2 \cos u_3 \tan u_4 \partial_{u_1} + u_2 \sin u_3 \tan u_4 \partial_{u_2} + \cos u_3 \cot u_4 \partial_{u_3} \\ &\quad + \sin u_3 \partial_{u_4}, \\ \mathcal{J}_2 &= u_1 \cos u_3 \tan u_4 \partial_{u_1} - u_1 \sin u_3 \tan u_4 \partial_{u_2} - \sin u_3 \cot u_4 \partial_{u_3} \\ &\quad + \cos u_3 \partial_{u_4}, \\ \mathcal{J}_3 &= u_2\partial_{u_1} - u_1\partial_{u_2} + \partial_{u_3}, \\ \mathcal{G}_1 &= \partial_{u_1}, \quad \mathcal{G}_2 = \partial_{u_2}, \quad \mathcal{G}_3 = \cos u_3 \tan u_4 \partial_{u_1} - \sin u_3 \tan u_4 \partial_{u_2},\end{aligned}$$

4. $$\begin{aligned}\mathcal{J}_1 &= F(\sec u_3)^2\partial_{u_1} + \cos u_2 \tan u_3 \partial_{u_2} - \sin u_2 \partial_{u_3}, \\ \mathcal{J}_2 &= (F(\sec u_3)^2 \tan u_2 + u_1 \sec u_2 \tan u_3)\partial_{u_1} + \sin u_2 \tan u_3 \partial_{u_2} \\ &\quad + \cos u_2 \partial_{u_3}, \\ \mathcal{J}_3 &= -u_1 \tan u_2 \partial_{u_1} - \partial_{u_2}, \\ \mathcal{G}_1 &= \partial_{u_1}, \quad \mathcal{G}_2 = \tan u_2 \partial_{u_1}, \quad \mathcal{G}_3 = \sec u_2 \tan u_3 \partial_{u_1}, \\ \mathcal{M} &= Q \sec u_2 \sec u_3 \partial_{u_1} + \partial_{u_4};\end{aligned}$$

5. $\mathcal{J}_1 = Q(\sec u_3)^2\partial_{u_1} + \cos u_2 \tan u_3 \partial_{u_2} - \sin u_2 \partial_{u_3}$,
$\mathcal{J}_2 = (Q(\sec u_3)^2 \tan u_2 + u_1 \sec u_2 \tan u_3)\partial_{u_1} + \sin u_2 \tan u_3 \partial_{u_2}$
$\quad + \cos u_2 \partial_{u_3}$,
$\mathcal{J}_3 = -u_1 \tan u_2 \partial_{u_1} - \partial_{u_2}$,
$\mathcal{G}_1 = \partial_{u_1}, \quad \mathcal{G}_2 = \tan u_2 \partial_{u_1}, \quad \mathcal{G}_3 = \sec u_2 \tan u_3 \partial_{u_1}$,
$\mathcal{M} = Q \sec u_2 \sec u_3 \partial_{u_1}$;

6. $\mathcal{J}_1 = \cos u_2 \tan u_4 \partial_{u_2} + (\cos u_2 + u_3 \sin u_2 \tan u_4)\partial_{u_3} - \sin u_2 \partial_{u_4}$,
$\mathcal{J}_2 = u_1 \sec u_2 \tan u_4 \partial_{u_1} + \sin u_2 \tan u_4 \partial_{u_2} + (\sin u_2$
$\quad - u_3 \cos u_2 \tan u_4)\partial_{u_3} + \cos u_2 \partial_{u_4}$,
$\mathcal{J}_3 = -u_1 \tan u_2 \partial_{u_1} - \partial_{u_2}$,
$\mathcal{G}_1 = \partial_{u_1}, \quad \mathcal{G}_2 = \tan u_2 \partial_{u_1}, \quad \mathcal{G}_3 = \sec u_2 \tan u_4 \partial_{u_1}$,
$\mathcal{M} = F \sec u_2 \sec u_4 \partial_{u_1} + G \cos u_4 \partial_{u_3} + \epsilon \partial_{u_5}$;

7. $\mathcal{J}_1 = \sin u_1 \tan u_3 \partial_{u_1} + R \sec u_3 \sin u_1 \partial_{u_2} + (Q \sin u_1 \sec u_3$
$\quad + \cos u_1)\partial_{u_3} + \epsilon \sin u_1 \sec u_3 \partial_{u_4}$,
$\mathcal{J}_2 = \cos u_1 \tan u_3 \partial_{u_1} + R \cos u_1 \sec u_3 \partial_{u_2} + (Q \cos u_1 \sec u_3$
$\quad - \sin u_1)\partial_{u_3} + \epsilon \cos u_1 \sec u_3 \partial_{u_4}$,
$\mathcal{J}_3 = \partial_{u_1}$,
$\mathcal{G}_1 = 0, \quad \mathcal{G}_2 = 0, \quad \mathcal{G}_3 = 0$,
$\mathcal{M} = \partial_{u_2}$;

8. $\mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}$,
$\mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}$,
$\mathcal{J}_3 = \partial_{u_1}$,
$\mathcal{G}_1 = 0, \quad \mathcal{G}_2 = 0, \quad \mathcal{G}_3 = 0$,
$\mathcal{M} = 0$;

9. $\mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - (\cos u_1 - \alpha \sin u_1 \sec u_2)\partial_{u_2}$
$\quad + \sin u_1 \sec u_2 \partial_{u_3}$,
$\mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + (\sin u_1 + \alpha \cos u_1 \sec u_2)\partial_{u_2}$
$\quad + \cos u_1 \sec u_2 \partial_{u_3}$,
$\mathcal{J}_3 = \partial_{u_1}$,

$\mathcal{G}_1 = 0, \quad \mathcal{G}_2 = 0, \quad \mathcal{G}_3 = 0,$
$\mathcal{M} = \epsilon\partial_{u_1};$

10. $\mathcal{J}_1 = 0, \quad \mathcal{J}_2 = 0, \quad \mathcal{J}_3 = 0,$
$\mathcal{G}_1 = 0, \quad \mathcal{G}_2 = 0, \quad \mathcal{G}_3 = 0,$
$\mathcal{M} = \epsilon\partial_{u_1}.$

Here f, g are arbitrary smooth functions of $u_6, \ldots, u_n$, F is an arbitrary smooth function of u_5, ..., u_n, R, Q are arbitrary smooth functions of u_4, ..., u_n, α is an arbitrary smooth function of $u_3, \ldots, u_n$ and $\epsilon = 0, 1$.

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